Review of works of Tokyo group on game-theoretic probability and some thoughts on basing probability theory on perfect information games

Akimichi Takemura

Univ. of Tokyo

June 21, 2010

Contents

History of Tokyo group

- Prof. Takeuchi had a series of expository articles in a Japanese general mathematics magazine from the end of 2002 based on S-V book.
- The articles became a book in 2004.
- Japanese translation of S-V book appeared in 2006.
- We (Kumon, Takemura, Takeuchi) started writing papers in 2005.
- Our 11 papers are listed on `http://www.probabilityandfinance.com/` with nice summaries!

  "Tokyo papers in chronological order"
- We managed to publish most of them (with much difficulty).

Features of works of Tokyo group

- Emphasis on explicit and single strategy (single: without infinite static mixture)
- Evaluation of order of divergence of capital processes
- Role of Kullback-Leibler information in the evaluation
- Proposal of new treatment of continuous-time process without non-standard analysis

List of Tokyo papers

- [1] "On a simple strategy weakly forcing the strong law of large numbers in the bounded forecasting game", Kumon and AT. (Aug05) [AISM, **60**, 801–812 (2008).]
- The article constructs an explicit strategy that weakly forces the strong law of large numbers in the bounded forecasting game with rate of convergence $O(\sqrt{\log n/n})$.

- Consider bounded forecasting game, where $x_n \in [-1, 1]$:

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n, \qquad x_n \in [-1, 1].$$

- Consider the strategy $M_n = c\bar{x}_{n-1}\mathcal{K}_{n-1},\ \ 0 < c < 1/2$, where $\bar{x}_{n-1}$ is the average of Reality's moves up to round $n-1$.
- Motivation: Skeptic's boldness depends on $\bar{x}_{n-1}$.
- With this strategy, Skeptic can weakly force the event

$$\limsup_n \frac{\sqrt{n}|\bar{x}_n|}{\sqrt{\log n}} \le 1.$$

List of Tokyo papers

- [2] "Game theoretic derivation of discrete distributions and discrete pricing formulas", AT and Taiji Suzuki. (Sep05) [*J. Japan Stat. Soc.*, **37**, 87–104 (2007).]
- The authors illustrate the generality of discrete finite-horizon game-theoretic probability protocols. The game-theoretic framework is advantageous because no a priori probabilistic assumption is needed.

This was a very introductory exposition on pricing formulas, such as Cox-Ross-Rubinstein formula. At this point I want to digress to tell you a "story" concerning pricing formula.

Betting on individual paths

- Suppose that you were a swindler and want to rip off some people. In the first week you send 1024 postcards (emails) to 1024 people. On one half (512) of the postcards, you write your prediction that the stock price will rise in the following week and on another half of the postcards you write that the stock price will fall.
- After one week you know that you were correct on half of the postcards. Now you divide 512 people, to whom you sent a correct prediction, into two groups of 256 people and you repeat the same thing as the first week.
- You repeat the procedure for 10 weeks. At the end of the 10'th week, there is exactly one person, to whom you have shown that you were correct for 10 consecutive weeks.
- Now you contact this person and ask him or her to invest in you.

Betting on individual paths

- You are a fund manager and you have 1024 people working for you. For simplicity suppose that the market is like the fair-coin game, so that for each weak you can either double your capital or lose your capital
- Because you have 1024 people working for you, you can assign each person to each distinct path for the coming 10 weeks. For example you ask one particular person to bet all his capital to the following path: (up, down, up, up, down, up, down, down, down, up).
- Now after 10 weeks, just one of 1024 people succeeds in multiplying his capital by the factor of 1024 and all other 1023 have nothing.
- At the beginning you divide your one dollar into $1/1024$ dollars and give each person $1/1024$ dollars. After 10 weeks you see that you can collect exactly 1 dollar from 1 of 1024 people. Recall that at the beginning you had 1 dollar. Therefore there is no uncertainty.
- This is because exactly one path out of 1024 paths realizes in 10 weeks.

Betting on individual paths

- Now consider a fair-coin game for N rounds and let $\xi = x_1 x_2 \dots x_N$ denote a path, where $x_n \in \{0, 1\}$.
- Consider pricing a path-dependent option $\eta(\xi)$. This option pays you $\eta(\xi)$ dollars at the end of N-th round when a particular path ξ has realized.
- As above suppose that you have 2^N people working for you. You assign a particular path $\xi^0 = (x_1^0 \dots x_N^0)$ to a particular person and give him the initial capital $\eta(\xi^0)/2^N = \eta(x_1^0 \dots x_N^0)/2^N$. Then after N rounds, just one person returns you $\eta(\xi)$ dollars, where ξ is now the realized path.

Betting on individual paths

- Therefore you have exactly replicated the option $\eta(\xi)$ with the initial capital

$$\sum_{\xi \in \{0,1\}^N} \frac{\eta(\xi)}{2^N}.$$

- It follows that this is the exact price of η. In particular when η depends only on the number of heads $S_N = x_1 + \cdots + x_N$, then the pricing formula becomes

$$\sum_{k=0}^{N} \binom{N}{k} \frac{1}{2^N} \eta(k),$$

  which corresponds to the Cox-Ross-Rubinstein formula.

- Note that the usual backward induction was not needed. [End of digression.]

List of Tokyo papers

- [3] "Capital process and optimality properties of Bayesian Skeptic in the fair and biased coin games", Kumon, AT and Takeuchi. (Oct05) [*Stochastic Analysis and Applications*, **26**, 1161–1180 (2008).]
- The article studies capital process behavior in the fair-coin and biased-coin games. A Bayesian strategy for Skeptic with a beta prior weakly forces the strong law of large numbers with rate of convergence $O(\sqrt{\log n/n})$. If Reality violates the law, then the exponential growth rate of the capital process is very accurately described in terms of Kullback divergence. The authors also investigate optimality properties of Bayesian strategies.

List of Tokyo papers

- Consider the coin-tossing game $x_n \in \{0, 1\}$ with price $1/2$:

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n(x_n - 1/2).$$

- Let $h_n = x_1 + \cdots + x_n$ be the number of heads and $t_n = n - h_n$ be the number of tails.
- Skeptic's Bayes strategy (uniform prior)

$$M_n = \mathcal{K}_{n-1} \frac{h_{n-1} - t_{n-1}}{n+1}$$

- Then $\mathcal{K}_n$ is explicitly written

$$\mathcal{K}_n = 2^n \frac{h_n! t_n!}{(n+1)!}$$

- Ville already had this. By Stirling's formula, we obtain Kullback divergence from this expression.

List of Tokyo papers

- [4] "Game-theoretic versions of strong law of large numbers for unbounded variables", Kumon, AT and Takeuchi. (Mar06) [*Stochastics*, **79**, 449–468 (2007).]

- The authors prove several versions of the game-theoretic strong law of large numbers in the case where Reality's moves are unbounded.

In S-V book, the quadratic hedge is considered. We proved SLLN under general family of hedges. In particular it should be noted that the hedge $|x_n|$ does not force SLLN:

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n + V_n(|x_n| - 1).$$

($|x_n|^{1+\epsilon}$ is OK)

List of Tokyo papers

- [5] "Implications of contrarian and one-sided strategies for the fair-coin game", Yasunori Horikoshi and AT. (Mar07) [*Stochastic Processes and their Applications*, **118**, 2125–2142 (2008).]

- The authors derive results on contrarian and one-sided strategies for Skeptic in the fair-coin game. For the strong law of large numbers, they prove that Skeptic can prevent the convergence from being faster than $n^{-1/2}$. They also derive a corresponding one-sided result.

List of Tokyo papers

- [6] "A new formulation of asset trading games in continuous time with essential forcing of variation exponent", Takeuchi, Kumon and AT. (Aug07) [*Bernoulli*, **15**, 1243–1258 (2009).]

- This article introduces a new formulation of continuous-time asset trading in the game-theoretic framework for probability. The market moves continuously but an investor trades at discrete times which can depend on the past path of the market.

The idea of η-step game was presented in Prof. Takeuchi's talk. It is a very simple but a powerful argument.

List of Tokyo papers

- [7] “Multistep Bayesian strategy in coin-tossing games and its application to asset trading games in continuous time”, Takeuchi, Kumon and AT. (Feb08) [To appear in *Stochastic Analysis and Applications*.]

- The article studies multistep Bayesian betting strategies in coin-tossing games in the framework of game-theoretic probability. By a countable mixture of these strategies, a gambler or an investor can exploit arbitrary patterns of deviations of nature's moves from independent Bernoulli trials. The authors apply their scheme to asset trading games in continuous time and derive the exponential growth rate of the investor's capital when the variation exponent of the asset price path deviates from two.

List of Tokyo papers

- In coin-tossing games, you calculated empirical conditional probabilities
$$P(x_n = j \mid x_{n-1} = i), \quad i, j = \pm 1.$$
- You can also consider higher-order empirical conditional probabilities
$$P(x_n = j \mid x_{n-1} = i_1, x_{n-2} = i_2), \quad i_1, i_2, j = \pm 1.$$
- These strategies efficiently exploit autocorrelations in Reality's move.
- If we apply this technique in η-step game, we can evaluate the growth rate of capital if the Hölder exponent of the price process deviates from $1/2$.

List of Tokyo papers

- [8] "The generality of the zero-one laws", by AT, Vovk and Shafer. (Mar08) [AISM, `doi:10.1007/s10463-009-0262-0`]
- The authors prove game-theoretic generalizations of some well known zero-one laws. Their proofs make the martingales behind the laws explicit, and their results illustrate how martingale arguments can have implications going beyond measure-theoretic probability.

This is my collaboration with Volodya and Glenn on zero-one laws. Actually I only supplied initial ideas.

List of Tokyo papers

- [9] "New procedures for testing whether stock price processes are martingales", Takeuchi, AT and Kumon. (Jul09). [*Computational Economics*, DOI:10.1007/s10614-010-9206-2 2010.]

- The authors propose procedures for testing whether stock price processes are martingales based on limit order type betting strategies. With high frequency Markov type strategies they find that martingale null hypotheses are rejected for many stocks traded on the Tokyo Stock Exchange.

We applied the idea of sequential tests to directly test whether several stock prices are martingales. If we adopt the idea of η-step game and only look at mutual independence of ups and downs between hitting times (ignore waiting times), then we can directly test martingaleness.

List of Tokyo papers

- [10] “Sequential optimizing strategy in multi-dimensional bounded forecasting games”, Kumon, AT and Takeuchi. (Nov09).
- The authors propose a sequential optimizing betting strategy in the multi-dimensional bounded forecasting game in the framework of game-theoretic probability. By studying the asymptotic behavior of its capital process, they prove a generalization of the strong law of large numbers. They also introduce an information criterion for selecting efficient betting items. These results are then applied to multiple asset trading strategies in discrete-time and continuous-time games. In conclusion they give numerical examples involving stock price data from the Tokyo Stock Exchange.

List of Tokyo papers

- Consider bounded forecasting game $x_n \in [-1, 1]$.
- At time n, maximize the following w.r.t. α:

$$\hat{\alpha}_n \ : \ \sum_{i=1}^{n-1}(1 + \alpha x_i) \ \rightarrow \ \max$$

- This gives the best "hindsight" constant for the constant-proportional ϵ-strategy in S-V book.
- Use the best constant $\hat{\alpha}_n$ until yesterday for today (round n).
- This strategy is flexible and show a good performance in various situations (theoretically and numerically).

List of Tokyo papers

- [11] "Sequential optimizing investing strategy with neural networks", Ryo Adachi and AT. arXiv:1002.2265v1 (Feb10).
- The authors propose an investing strategy based on neural network models combined with ideas from game-theoretic probability. Their strategy uses parameter values of a neural network with the best performance until the previous round (trading day) for deciding the investment in the current round. They compare their proposed strategy with various strategies including a strategy based on supervised neural network models and show that their strategy is competitive with other strategies.

List of Tokyo papers

- The idea of SOS was combined with neural network models.
- In neural network models, people use highly non-linear functions $f(x;\theta)$, where $x \in [-1,1]^k$ is the input to the neural network, θ is a parameter vector of the network and $f \in [-1,1]$ is the output.
- At time n, maximize the following w.r.t. θ:

$$\hat{\theta}_n \ : \ \sum_{i=1}^{n-1}(1 + f(x_{i-1},\ldots,x_{i-k},\theta)x_i) \ \rightarrow \ \max$$

- Use $f(x_{n-1},\ldots,x_{n-l},\hat{\theta}_n)$ for betting today. This gives a flexible Markov strategy for bounded forecasting game.

Some thoughts on game-theoretic probability

- Motivation for working on GTP
- No need for measure theory?
- Doubling strategy and absolute continuity of measures
- Non-negative martingales and likelihood ratios
- Example : protocol for $N(0,1)$, $N(\mu,1)$.
- Reality's deterministic strategy

Motivation for working on GTP

- There are lots of new problems in GTP.
- We can write papers. However it is difficult to publish them.
- Motivation: more fundamental questions on probability and statistics.
- There is a possibility to change foundations of probability theory, thus influencing many fields using probability

No need for measure theory?

- GTP can be based on upper prices.
- Upper prices are outer measures.
- When we learn measure theory, we need Caratheodory's extension theorem to define measurability from outer measures. Is this whole extension theorem needed?
- Extension theorem is not constructive anyway.
- Example: is MLE measurable?
- If we want to show that certain events have probability zero, we only need outer measure. Why restrict ourselves to measurable events?

Doubling strategy and absolute continuity of measures (1)

- Consider coin tossing. $X_n = 0$ or 1 are i.i.d. Bernoulli random variables.
- If they not degenerate at 0, i.e. if $0 < P(X_n = 1) = p$, then doubling strategy works.
- If we consider Brownian motion, then in arbitrary small time interval, we can perform infinite number of coin tosses. Therefore doubling strategy has to be excluded in some way. I do not believe that standard textbooks on mathematical finance treat this adequately.

Doubling strategy and absolute continuity of measures (2)

- As probability measures, the case of $p > 0$ and the case of $p = 0$ have one-sided domination. They are not mutually singular, nor mutually absolutely continuous.
- As long as we only observe a finite sequence of consecutive 0's, we can not tell whether $p = 0$ or $p > 0$.
- Volodya once taught me in his email, that doubling strategy in no issue in GTP.
- Indeed, because Reality does not have to follow a probability and can always choose a move adversarial to Skeptic, doubling strategy is no issue in GTP.

Non-negative martingales and likelihood ratios

As a standard textbook material I will confirm that in measure-theoretic framework the following two things are equivalent.

1. Non-negative martingales with expected value 1.
2. Likelihood ratios

I will discuss game-theoretic interpretations later.

Non-negative martingales and likelihood ratios

Martingale $\Rightarrow$ LR

- Let $\mathcal{F}_n$, $n = 0, 1, 2, \ldots$ be a filtration.
- Let $\mathcal{F} = \mathcal{F}_\infty$ be the smallest σ-field containing them.

- Fix a probability measure P on $\mathcal{F}$ and let M_n, $n = 0, 1, 2, \ldots$ be a non-negative martingale under P with $E(M_n) = 1$, $\forall n$.
- Define Q_n on $\mathcal{F}_n$ by

$$Q_n(A) = \int_A M_n dP, \quad A \in \mathcal{F}_n. \tag{1}$$

- Then $Q_n(\Omega) = \int_\Omega M_n dP = E(M_n) = 1$ and Q_n is a probability measure on $\mathcal{F}_n$.

Non-negative martingales and likelihood ratios

- Furthermore $M_n = dQ_n/dP$.
- We want to show that Q_n, $n = 1, 2, \dots$ are consistent:

$$\text{For } A \in \mathcal{F}_n, \quad Q_n(A) = Q_{n+1}(A).$$

- This is checked as follows: If $A \in \mathcal{F}_n$ then $A \in \mathcal{F}_{n+1}$ and

$$\begin{aligned} Q_{n+1}(A) &= \int_A M_{n+1} dP = E(I_A M_{n+1}) = E(E(I_A M_{n+1}|\mathcal{F}_n)) \\ &= E(I_A E(M_{n+1}|\mathcal{F}_n)) = E(I_A M_n) = Q_n(A), \end{aligned}$$

where E denotes the expected value under P.

Non-negative martingales and likelihood ratios

Conversely, LR $\Rightarrow$ Martingale

- Let $Q_1, Q_2, \ldots$ be a consistent family of probability distributions on $\mathcal{F}_n$, $n = 0, 1, 2, \ldots,$, such that each Q_n is absolutely continuous with P.
- Define
$$M_n = \frac{dQ_n}{dP}.$$
- Then
$$E(M_n) = \int_\Omega \frac{dQ_n}{dP} dP = \int_\Omega dQ_n = Q_n(\Omega) = 1.$$

Non-negative martingales and likelihood ratios

- Furthermore we show $E(M_{n+1}|\mathcal{F}_n) = M_n$. It suffices to check that for any $A \in \mathcal{F}_n$

  $$\int_A M_n dP = \int_A M_{n+1} dP.$$

  However this is equivalent to the consistency condition $Q_n(A) = Q_{n+1}(A)$.
- Hence the sequence of likelihood ratios is a non-negative martingale with expected value 1.

Non-negative martingales and likelihood ratio

- We have now shown that there is a one-to-one correspondence between a non-negative martingale M_n under P and the a sequence of consistent probability measures Q_n absolutely continuous w.r.t. P.
- From GTP, the capital process for any prudent strategy is a non-negative martingale with expected value 1 under any risk neutral probability measure.

Non-negative martingales and likelihood ratio

- However not all non-negative martingales with expected value 1 can be realized as a capital process. It depends on how rich is the move space of Skeptic.
- If the game is "complete" or "non-redundant", such as the horse race game, then the converse is true.
- Therefore the following question is of theoretical interest: Given a measure-theoretic non-negative martingale with expected value 1, set up a protocol of a game, such that the martingale can be realized as a capital process in the game.

Example : Protocol for $N(0,1)$

- What is the game for "$X_1, X_2, \ldots$ *i.i.d.* $N(0,1)$"?
- Let $E^{N(0,1)}$ denote the expected value under $N(0,1)$.
- Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} \mid E^{N(0,1)}(f) = 0\}$. (can be a much smaller set)
- Following "Generalized Coin Tossing" on p.181 of S-V book, we can write the protocol as follows

  $\mathcal{K}_0 = 1$
  FOR $n = 1, 2, \ldots$
      Skeptic announces $f_n \in \mathcal{F}$.
      Reality announces $x_n \in \mathbb{R}$
      $\mathcal{K}_n := \mathcal{K}_{n-1} + f_n(x_n)$.
  END FOR
- Under this protocol, Reality's moves $x_1, x_2, \ldots$ are not distinguishable from i.i.d. $N(0,1)$ variables.

Example : Protocol for $N(\mu, 1)$?

- Can we force that Reality's moves are like i.i.d. observations from $N(\mu, 1)$ for some μ? (I am not sure, but relevant to statistics).
-
$$\mathcal{K}_n := \mathcal{K}_{n-1} + g_n(x_n; \bar{x}_{n-1}) + f_n(x_n; \bar{x}_{n-1}),$$
where g_n is a term which somehow forces convergence of $\bar{x}_n$ and f_n is any function with $E^{N(\bar{x}_{n-1},1)}(f_n) = 1$.
- I am not sure if the following g_n works:
$$g_n = M_n(x_n - \bar{x}_{n-1}),$$
- or maybe
$$g_n = M_n((x_n - \bar{x}_{n-1})^2 - 1).$$

Reality's non-randomized strategy

- In S-V book, Reality's strategy is discussed (only) in Section 4.3.
- Unfortunately it is randomized.
- Consider the unbounded protocol with quadratic hedge ($v_n \geq 0$ is announced by Forecaster).

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n x_n + V_n(x_n^2 - v_n).$$

- The second part of Theorem 4.1 of S-V says that Reality can force

$$\sum_{n=1}^{\infty} \frac{v_n}{n^2} = \infty \ \Rightarrow \ \big(\bar{x}_n \text{ does not converge to } 0\big)$$

Reality's non-randomized strategy

- Reality's randomized strategy: if $v_n < n^2$, then choose

$$x_n := \begin{pmatrix} n \\ -n \\ 0 \end{pmatrix} \text{ with probability } \begin{pmatrix} v_n/(2n^2) \\ v_n/(2n^2) \\ 1 - v_n/n^2 \end{pmatrix}$$

and if $v_n \geq n^2$, then choose

$$x_n := \begin{pmatrix} \sqrt{v_n} \\ -\sqrt{v_n} \end{pmatrix} \text{ with probability } \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Reality's non-randomized strategy

- The following strategy τ (due to Volodya) is deterministic:
  - If $\mathcal{K}_{n-1} - |M_n|n + V_n(n^2 - v_n) \leq 1$, choose $x_n = \pm n$, with the sign chosen such that $M_n x_n \leq 0$.
  - Otherwise choose $x_n = 0$.
- Obviously, under τ, $\mathcal{K}_n \leq 1$, whatever the moves of Forecaster and Skeptic are.
- When $\sum_{n=1}^{\infty} v_n/n^2 = \infty$, either Reality can choose $\pm n$ infinitely often, so that $\bar{x}_n$ does not converge to 0, or there is some n such that $\mathcal{K}_n < 0$. So Reality wins.
- I have some extensions of the above result to 0-1 laws in coin-tossing games. But I am not satisfied with my results yet.

Summary

- I have explained some works of Tokyo group.
- I have talked about some thoughts on GTP. Mainly I touched upon the problem of setting up an appropriate game, given a statistical model such as $N(0,1)$.

Thank you for your attention!