Zero-sum two person game, Kullback-Leibler information, sequential hypothesis testing, in relation to Shafer-Vovk theory

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S-V theory and 0-sum 2-person game

1.1 Formulation

The unit zero-sum two-person game:

- **Pay-off matrix** $A = \{a_{ij}\} : k \times m$**.**
- **The first player: Skeptic**
- **The second player: Reality**
- **There are** k **items Skeptic can bet and** m **outcomes Reality can choose.**
- **The value of the game** $v^* = v^*(A)$**.**

Protocol of the betting game

Protocol:

$\mathcal{K}_0 = 1$

FOR $n = 1, 2, \dots$

  Skeptic announces $M_{ni} \geq 0,\ i = 1, \dots, k$**.**

  Reality announces $j_n \in \{1, \dots, m\}$**.**

  $\mathcal{K}_n = \mathcal{K}_{n-1} + \sum_{i=1}^{k} M_{ni} x_{ni}$**, where** $x_{ni} = a_{ij_n}$**.**

END FOR

(Since $M_{ni} \geq 0$**, we are considering a one-sided game.)**

- **The set of optimal strategies for Skeptic in the unit game:**

$$Q^* = \{\boldsymbol{\alpha}\},\ \boldsymbol{\alpha} = (\alpha_1^*, \dots, \alpha_k^*)',\ \alpha_i^* \geq 0, \sum_{i=1}^{k} \alpha_i^* = 1.$$

- **The set of optimal strategies for Reality:**

$$P^* = \{\boldsymbol{p}\},\ \boldsymbol{p} = (p_1^*, \dots, p_m^*)',\ p_j^* \geq 0, \sum_{j=1}^{m} p_j^* = 1.$$

- **The value v^* of the game:**

$$\textbf{For } \boldsymbol{\alpha}^* \in Q^*,\ \sum_{i=1}^{k} a_{ij}\alpha_i^* \geq v^*,\ \forall j \in \{1,\ldots,m\} \quad (1)$$

$$\textbf{For } \boldsymbol{p}^* \in P^*,\ \sum_{j=1}^{m} a_{ij}p_j^* \leq v^*,\ \forall i \in \{1,\ldots,k\} \quad (2)$$

- **If $v^* > 0$, then Skeptic can make $\mathcal{K}_n \to \infty$ in one step, by (1).**

- **If $v^* < 0$, then Reality can make $\mathcal{K}_n \leq 0$ as long as Skeptic keeps betting positive amount, which is bounded away from 0.**

- **We call the betting game *fair* if $v^* = 0$.**

1.2 Some definitions on the unit game

From now on we assume $v^* = 0$.

Definition 1 *The unit game is* regular *if for every j, there exists $\boldsymbol{p}^* \in P^*$ such that $p_j^* > 0$.*

Proposition 1 *If the unit game is non-regular (i.e. if for some j_0, $p_{j_0}^* = 0, \forall \boldsymbol{p}^* \in P^*$), then there exists $\boldsymbol{\alpha}^* \in Q^*$ such that*

$$\sum_{i=1}^{k} a_{ij_0}\alpha_i^* > 0, \qquad \sum_{i=1}^{k} a_{ij}\alpha_i^* \geq 0, \forall j.$$

Proposition 2 *If the unit game is regular, then there exists* $\boldsymbol{p}^* \in P^*$ *such that* $p_j^* > 0, \forall j$.

Definition 2 *The unit game is* non-redundant, *if it is regular and* $\boldsymbol{p}^* \in P^*$ *is unique.*

(corresponds to "complete market" in mathematical finance)

Proposition 3 *If the unit game is non-redundant with* $\{\boldsymbol{p}^*\} = P^*$, *then* $\sum_{i=1}^k a_{ij}\alpha_i^* = 0, \forall j, \forall \boldsymbol{\alpha}^* \in Q^*$.

Definition 3 *The j_0-th outcome is* weakly dominated *if these is $\boldsymbol{p}$ such that $p_j \geq 0$, $\forall j$, $\sum_j p_j = 1$, $p_{j_0} = 0$, and*

$$\sum_{j=1}^{m} a_{ij} p_j \leq a_{ij_0}, \ \forall i. \tag{3}$$

The j_0-th outcome is strongly dominated *if the inequality in (3) is strict for all i.*

Proposition 4 *If the unit game is regular, no outcome for Reality is strongly dominated, and when it is non-redundant, no outcome for Reality is weakly dominated.*

Proposition 5 *If the j_0-th outcome for Reality is weakly dominated, the value of the unit game does not change when the j_0-th outcome is deleted from the game.*

Definition 4 *The i_0-th item Skeptic is weakly dominated if there is $\{\alpha_i\}$ such that $\alpha_i \geq 0$, $\forall i$, $\sum_i \alpha_i = 1$, $\alpha_{i_0} = 0$ and*

$$\sum_{i=1}^{k} a_{ij}\alpha_i \geq a_{i_0 j}, \ \forall j.$$

It is strongly dominated if the inequalities are strict for all j.

1.3 Price of an additional item

- **Let A be a pay-off matrix with $v^*(A) = 0$.**
- **Let $\boldsymbol{b} = \{b_j\}$ be an additional item for Skeptic. Assume $b_j > 0, \exists j$.**
- **For $\pi > 0$ define an augmented payoff matrix**

$$\bar{A} = \begin{pmatrix} & A & \\ \frac{b_1}{\pi} - 1 & \dots & \frac{b_m}{\pi} - 1 \end{pmatrix}$$

Definition 5 *π is a* proper price *of $\boldsymbol{b}$, if $v^*(\tilde{A}) = 0$ and there is an optimal strategy for Skeptic in $\tilde{A}$ for which $\alpha_{k+1} > 0$.*

Proposition 6 *$v^*(\tilde{A}) = 0$ if and only if*

$$\pi \geq \underline{\pi}^* = \inf_{\boldsymbol{p}^* \in P^*} \sum_j b_j p_j^*.$$

(inf **and** sup **are attained because our setup is finite-dimensional.)**

Proposition 7 *Let π be the proper price.*

$$\pi \leq \bar{\pi}^* = \sup_{\boldsymbol{p}^* \in P^*} \sum_j b_j p_j^*.$$

Proposition 8 $\bar{\pi}^*$ *can be expressed as the upper expectation*

$$\bar{\pi}^* = \min\{\pi \mid \exists\{\alpha_i \geq 0\} \text{ s.t.} \sum_{i=1}^{k} a_{ij}\alpha_i \geq b_j - \pi, \forall j\}.$$

1.4 Expanded form of multi-stage betting game

Consider the N-step betting game in the multiplicative form $\alpha_{ni} = M_{ni}/\mathcal{K}_{n-1}$:

$\mathcal{K}_0 = 1$.
FOR $n = 1, \ldots, N$**.**
  Skeptic announces $\alpha_{ni} \geq 0,\ i = 1, \ldots, k$**.**
  Reality announces $j_n \in \{1, \ldots, m\}$**.**
  $\mathcal{K}_n = \mathcal{K}_{n-1}(1 + \sum_{i=1}^{k} \alpha_{ni} x_{ni})$**.**
END FOR

- **Skeptic chooses**

$$I = \{i_1, i_2(j_1), i_3(j_1, j_2), \dots, i_N(j_1, \dots, j_{N-1})\}$$

- **Reality chooses**

$$J = \{j_1, j_2, \dots, j_N\}$$

- **Expanded payoff matrix $\{\tilde{A}_{IJ}\}$:**

$$\tilde{A}_{IJ} = (1+a_{i_1 j_1})(1+a_{i_2(j_1)j_2}) \dots (1+a_{i_N(j_1,\dots,j_{N-1})j_N}) - 1$$

- **Optimal strategies: $\{\boldsymbol{\alpha}_I^*\} \in \tilde{Q}^*, \{\boldsymbol{p}_J^*\} \in \tilde{P}^*$.**
- **If $v^*(A) = 0$ then $v^*(\{\tilde{A}_{IJ}\}) = 0$.**

Theorem 1 *Strategies for two players are optimal if and only if all the conditional strategies (conditional probabilities) are optimal for the unit game, i.e.*

$$\{\tilde{\boldsymbol{\alpha}}_I^*\} \in \tilde{Q}^* \;\Leftrightarrow\; \{\alpha_{i_k(j_1,\ldots,j_{k-1})}\} \in Q^*, \quad \forall (j_1,\ldots,j_{k-1}), \forall k.$$

$$\{\tilde{\boldsymbol{p}}_J^*\} \in \tilde{P}^* \;\Leftrightarrow\; \{p_{j_k(j_1,\ldots,j_{k-1})}\} \in P^*, \quad \forall (j_1,\ldots,j_{k-1}), \forall k.$$

Additional item ("derivative") in N-step betting game. $\phi(J) = \phi(j_1, \dots, j_N)$

- **Let**

$$\bar{\pi}^*_\phi = \sup_{\{\boldsymbol{p}^*_J\} \in \tilde{P}^*} E_{\{\boldsymbol{p}^*_J\}}(\phi(J)), \qquad \underline{\pi}^*_\phi = \inf_{\{\boldsymbol{p}^*_J\} \in \tilde{P}^*} E_{\{\boldsymbol{p}^*_J\}}(\phi(J))$$

- **These are calculated by backward induction.**

Definition 6

$$\bar{\pi}^*_\phi(j_1, \dots, j_k) = \sup_{\boldsymbol{p}_{k+1}, \dots, \boldsymbol{p}^*_N \in P^*} E(\phi(j_1, \dots, j_k, j_{k+1}, \dots, j_N))$$
$$\underline{\pi}^*_\phi(j_1, \dots, j_k) = \inf_{\boldsymbol{p}_{k+1}, \dots, \boldsymbol{p}^*_N \in P^*} E(\phi(j_1, \dots, j_k, j_{k+1}, \dots, j_N))$$

- **Then**

$$\bar{\pi}^*_\phi(j_1,\ldots,j_k) = \sup_{\boldsymbol{p}^*_{k+1}\in P^*} E(\bar{\pi}^*_\phi(j_1,\ldots,j_k,j_{k+1}))$$

$$\underline{\pi}^*_\phi(j_1,\ldots,j_k) = \inf_{\boldsymbol{p}^*_{k+1}\in P^*} E(\bar{\pi}^*_\phi(j_1,\ldots,j_k,j_{k+1}))$$

- **The case of non-redundant unit game.**

  If the unit game A is non-redundant with the unique optimal strategy $\boldsymbol{p}^*$, then the unique optimal strategy $\tilde{\boldsymbol{p}}^*$ for the expanded game is the N-fold direct product (namely, i.i.d.) $\tilde{\boldsymbol{p}}^* = \boldsymbol{p}^* \times \cdots \times \boldsymbol{p}^*$ of $\boldsymbol{p}^*$ and

$$\bar{\pi}^*_\phi = \underline{\pi}^*_\phi = E_{\tilde{\boldsymbol{p}}^*}(\phi(J)) = \pi^*_\phi.$$

Sequence of N-step games $\quad (N \to \infty)$

- **Consider a sequence of N-step games with the unit game $A_N = A/\sqrt{N} = \{a_{ij}/\sqrt{N}\}$**
- **Let**

$$\phi(j_1, \ldots, j_N) = \psi\Big(\sum_{i=1}^{k} c_i \sum_{n=1}^{N} a_{ij_n}/\sqrt{N}\Big),$$

where $\psi \in C^3(\mathbb{R}^1)$.

- **Let A be non-redundant. Then as $N \to \infty$**

$$\pi_\phi^* \to \int_{-\infty}^{\infty} \psi(u) \frac{1}{\sqrt{2\pi V}} \exp(-\frac{u^2}{2V}) du,$$

where $V = \sum_{i,i'} c_i c_{i'} \sum_j a_{ij} a_{i'j} p_j^*$.

- **Suppose that A is not non-redundant ("incomplete case"). Let**

$$\bar{V} = \sup_{\boldsymbol{p}^* \in P^*} \sum_{i,i'} c_i c_{i'} \sum_j a_{ij} a_{i'j} p_j^*$$

$$\underline{V} = \inf_{\boldsymbol{p}^* \in P^*} \sum_{i,i'} c_i c_{i'} \sum_j a_{ij} a_{i'j} p_j^*$$

Then

$$\begin{matrix} \bar{\pi}^*_\phi \\ \underline{\pi}^*_\phi \end{matrix} \rightarrow \int_{-\infty}^{\infty} \psi(u) \frac{1}{\sqrt{2\pi\bar{V}}} \exp(-\frac{u^2}{2\bar{V}}) du, \ \textbf{if } \psi \textbf{ is } \begin{matrix} \textbf{convex} \\ \textbf{concave} \end{matrix}$$

$$\begin{matrix} \underline{\pi}^*_\phi \\ \bar{\pi}^*_\phi \end{matrix} \rightarrow \int_{-\infty}^{\infty} \psi(u) \frac{1}{\sqrt{2\pi\underline{V}}} \exp(-\frac{u^2}{2\underline{V}}) du, \ \textbf{if } \psi \textbf{ is } \begin{matrix} \textbf{convex} \\ \textbf{concave} \end{matrix}$$

If ψ is *neither convex nor concave*, we have no explicit expression for the limiting formula for the upper and lower prices.

- **Define**

$$\bar{\pi}(x, \frac{n}{N}) = \sup_{\boldsymbol{p}^*_{n+1},\ldots,\boldsymbol{p}^*_N} E(\psi(x + \sum_i c_i \sum_{l=n+1}^{N} a_{ij_l}/\sqrt{N})).$$

- **Under some regularity conditions**

$$\bar{\pi}(x, n/N) \to \pi^*(x,t), \quad 0 \le t \le 1.$$

- **$\pi^*(x,t)$ is the solution of the following partial differential equation**

$$\frac{\partial \pi^*}{\partial t} = \frac{1}{2}\sigma_x^2 \frac{\partial^2 \pi^*}{\partial x^2}, \text{ where } \sigma_x^2 = \begin{cases} \bar{V} & \textbf{if } \frac{\partial^2 \pi^*}{\partial x^2} \geq 0 \\ \underline{V} & \textbf{if } \frac{\partial^2 \pi^*}{\partial x^2} < 0 \end{cases}$$

- **This partial differential equation is sometimes called "Black-Scholes-Barenblatt equation". When $\underline{V} = 0$, the solution for the equation exists in the sense of viscosity solution.**

[this is discussed in a manuscript in preparation.]

2. Kelly's strategy and Kullback-Leibler information

Kelly's strategy and KL-information

- **If Reality adopts a non-optimal strategy, then Skeptic can exploit the error.**
- **Assume that Reality's strategy $\boldsymbol{p}_0 = \{p_j^0\}$ is not in P^*. (common for each round).**
- **Skeptic chooses Kelly's strategy which maximizes**

$$E_{\boldsymbol{p}_0}(\log \mathcal{K}_1) = \sum_{j=1}^{m} p_j^0 \log(1 + \sum_{i=1}^{k} \alpha_i p_j^0).$$

- **Assume that the game is regular and** $p_j^0 > 0$ **for all** j**. Then Skeptic's best strategy** α_i^* **satisfies**

$$\sum_{j=1}^{m} \frac{a_{ij}}{1 + \sum_{h=1}^{k} \alpha_h^* a_{hj}} p_j^0 \le 0, \ \forall i, \ \textbf{and}$$

$$= 0 \quad \textbf{if } \alpha_i^* > 0$$

- **By defining**

$$p_j^* = \frac{1}{1 + \sum_{h=1}^{k} \alpha_h^* a_{hj}} p_j^0$$

we have

$$\sum_{j=1}^{m} a_{ij} p_j^* \le 0, \ \forall i, \quad \sum_{j=1}^{m} p_j^* = 1.$$

- **Therefore $\{p_j^*\} \in P^*$.**

Theorem 2 $\log \mathcal{K}_1 = 1 + \sum_h \alpha_h^* a_{hj} = \log(p_j^0/p_j^*)$ *and*

$$E_{\boldsymbol{p}_0}(\log \mathcal{K}_1) = D(\boldsymbol{p}_0 \mid \boldsymbol{p}^*) = \inf_{\boldsymbol{p}\in P^*} D(\boldsymbol{p}_0 \mid \boldsymbol{p}),$$

where $D(\boldsymbol{p}_1 \mid \boldsymbol{p}_2) = \sum_j p_j^1 \log \frac{p_j^1}{p_j^2}$ *is KL divergence. If* $p_j^* = 0$ *for some* j *in a regular game,* $\sup \mathcal{K}_1 = \infty$.

- **For N-step game, suppose that Reality adopts $\tilde{\boldsymbol{p}}^0$. Define $\alpha_{in}^* = \alpha_{i(j_1 \dots j_{n-1})}^*$ by**

$$\sum_{j=1}^{m} \frac{a_{ij}}{1 + \sum_{h=1}^{k} \alpha_{hn}^* a_{hj}} p_{j(j_1 \dots j_{n-1})}^0 \leq 0, \ \forall i$$

Theorem 3

$$E_{\tilde{\boldsymbol{p}}^0}(\log \mathcal{K}_N) = D(\tilde{\boldsymbol{p}}^0 \mid \tilde{\boldsymbol{p}}^*) = \inf_{\tilde{\boldsymbol{p}} \in \tilde{P}^*} D(\tilde{\boldsymbol{p}}^0 \mid \tilde{\boldsymbol{p}}).$$

When $\tilde{\boldsymbol{p}}_0 = (\boldsymbol{p}_0)^N$ *(N-fold direct product),*

$$E_{\tilde{\boldsymbol{p}}^0}(\log \mathcal{K}_N) = N D(\boldsymbol{p}_0 \mid \boldsymbol{p}^*) = N \inf_{\boldsymbol{p} \in P^*} D(\boldsymbol{p}_0 \mid \boldsymbol{p}).$$

The case of unknown $\boldsymbol{p}_0$ (still in usual statistical setting)

- **Let** $\hat{p}_{nj}$ **be an estimate of** p_j^0 **at round** n**.**
- **Define** $\hat{\alpha}_{ni}$ **and** $\hat{p}^*_{nj}$ **by**

$$(1 + \sum_h \hat{\alpha}_{nh} a_{hj}) \hat{p}^*_{nj} = \hat{p}_{nj}.$$

- **Then**

$$\begin{aligned} \log \mathcal{K}_N &= \sum_{n=1}^N \log \frac{\hat{p}_{nj_n}}{\hat{p}^*_{nj_n}} \\ &= \sum_{n=1}^N \log \frac{\hat{p}_{nj_n}}{p^*_j} - \sum_{n=1}^N \log \frac{\hat{p}^*_{nj_n}}{p^*_j}. \end{aligned}$$

- **Bayes estimator**

$$\hat{p}_{nj} = \frac{m_{nj} + c}{n + mc}, \qquad m_{nj} = \#\{j_l = j \mid 1 \le l \le n{-}1\}, c > 0.$$

-

$$\sum_{n=1}^{N} \log \hat{p}_{nj} = \sum_{j=1}^{m} \log \Gamma(m_{Nj} + c) - \log \Gamma(N + mc) - m \log \Gamma(c) + \log \Gamma(mc).$$

- **By Stirling's formula, we can prove the following game-theoretic theorem.**

Theorem 4

$$\log \mathcal{K}_N = N \sum_{j=1}^{m} \hat{p}_{Nj} \log \frac{\hat{p}_{Nj}}{\hat{p}_j^*} - \frac{m}{2} \log N - \sum_{n=1}^{N-1} \log \frac{\hat{p}_{nj_n}^*}{p_{j_n}^*} + O(1)$$
$$= N D(\hat{\boldsymbol{p}} \mid \boldsymbol{p}^*) - O(\log N)$$

and Skeptic can force Reality to make ($\epsilon > 0$ arbitrary)

$$\limsup_N N^{-\frac{1}{2}-\epsilon} \sum_{n=1}^{N} a_{ij_n} \le 0, \quad \forall i.$$

[Partly discussed in papers No.3 and No.10]

- **Let** $\{b_j\}$ **an additional item in the unit game with price** π**.**
- **Write** $a_{k+1,j} = \frac{b_j}{\pi} - 1$**.**
- **The best strategy** $\{\alpha_i^*\}$**,** $i = 1, \ldots, k+1$**, for Skeptic satisfies**

$$\sum_j \frac{a_{ij}}{1 + \sum_{h=1}^{k+1} \alpha_h^* a_{hj}} p_j^0 \leq 0, \quad i = 1, \ldots, k+1.$$

- **Putting** $(1 + \sum_{h=1}^{k+1} a_{hj}\alpha_h^*)p_j^* = p_j^0$**, we have** $\sum_j a_{ij}p_j^* \leq 0$**,** $i = 1, \ldots, k$ **and**

$$\sum_j b_j p_j^* \leq \pi.$$

- **Define $\boldsymbol{p}^*$ by $D(\boldsymbol{p}^0 \mid \boldsymbol{p}^*) = \inf_{\boldsymbol{p}\in P^*} D(\boldsymbol{p}^0 \mid \boldsymbol{p}^*)$. Then**

$$\sum_j b_j p_j^* < \pi \Rightarrow \alpha_{k+1}^* = 0,$$

$$\sum_j b_j p_j^* > \pi \Rightarrow \alpha_{k+1}^* > 0.$$

3. Sequential hypothesis testing

Sequential hypothesis testing

- **The null hypothesis:**

$$H \ : \ \boldsymbol{p} \in P^*.$$

- **Under H, $\mathcal{K}_n$ is a positive martingale.**
- **Let Σ be a stopping rule and let N be the associated stopping time.**
- **$\Sigma(j_1, \ldots, j_n) = 0$ or 1. As long as $\Sigma = 0$ Skeptic continues gambling and once $\Sigma(j_1, \ldots, j_N) = 1$ he stops gambling and retrieves $\mathcal{K}_N$.**

By the optional stopping theorem we have

Theorem 5 *Under the null hypothesis*

$$E(\mathcal{K}_N) = \mathcal{K}_0, \qquad P^*(\mathcal{K}_N/\mathcal{K}_0 \geq c) \leq \frac{1}{c}, \quad c > 0.$$

If we reject H *when* $\mathcal{K}_N/\mathcal{K}_0 \geq c$*, then the level of significance* α *satisfies* $\alpha \leq 1/c$*.*

- **With Kelly's strategy against** $\boldsymbol{p}_0$

$$\frac{\mathcal{K}_N}{\mathcal{K}_0} = \prod_{n=1}^{N} \frac{p^0_{nj_n}}{p^*_{nj_n}} = \mathbf{LR}.$$

Wald's SPR test

- **Let** $\Sigma = 0$ **for** $d < \mathcal{K}_n < c$**, reject** H **if** $\mathcal{K}_N \geq c$**, and accept** H **if** $\mathcal{K}_N \leq d$**.**

- **This test is identical with the SPR test and has the optimality of SPR test.**

- **For any capital process $\mathcal{K}_n$ with $\mathcal{K}_0 = 1$, a sequential test is obtained by:**
  - **Stop as soon as $\mathcal{K}_n \geq c$ or $\mathcal{K}_n \leq d$ happens.**
  - **In the former case reject H and in the latter case accept H.**
  - **The approximate level of significance α is**

$$\alpha \sim \frac{1-d}{c-d}.$$

Example: multinomial distribution.

- $X_1, X_2, \ldots,$ **i.i.d.** $P(X_n = j) = p_j$**,** $j = 1, \ldots, k$**.**
- $H : p_j = p_j^*$**.**

- **Corresponding game:** $A = \{a_{ij}\}$**,** $1 \le i, j \le k$**.**

$$a_{ii} = \frac{1}{p_i^*} - 1, \quad a_{ij} = -1, \ i \neq j.$$

- **Consider a test based on Bayes Kelly's strategy**

$$\hat{\alpha}^*_{n,i} = \frac{m_{ni} + c}{n + mc}$$

- **With** $c = 1$**, the test statistic is given as**

$$\mathcal{K}_n = \frac{n!}{(n+k)!} \Big/ \Big(\frac{n!}{\prod_j m_{nj}!} \prod_j (p_j^*)^{m_{nj}} \Big).$$

4. Asset trading games in continuous time

4.1 Formulation

4.2 η-step strategy [paper No.10, 11]

4.3 Strategies in induced η-step games

Asset trading games in continuous time

4.1 Formulation

- $S(t)$, $0 \le t \le T$: **the price of a tradable asset in continuous time.**
- $\mathrm{T.V.}(S)$: **total variation of** S **on** $[0, T]$.

$$\mathrm{T.V.}(S) = \sup_{0=t_0<t_1<\cdots<t_N<t_{N+1}=T} \sum_{i=0}^{N} |S(t_{i+1}) - S(t_i)|$$

(this could be infinite)

Protocol:

- **Reality chooses a continuous positive function S with $\mathrm{T.V.}(S) \geq A$.** **($A > 0$: given)**
- **Skeptic chooses trading times and number of assets to hold.**
  - **He chooses $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = T$, where t_n can depend on $\{S(t) \mid t \leq t_n\}$.**
  - **At time t_{n-1} he also chooses the number of units M_n of the asset to hold, depending on $\{S(t) \mid t \leq t_{n-1}\}$.**
- **Then the capital $\mathcal{K}_n$ at time t_n is written as**

$$\mathcal{K}_n = \mathcal{K}_{n-1} + M_n(S(t_n) - S(t_{n-1}))/S(t_{n-1}).$$

4.2 η-step strategy

- **Choose** $t_0 = 0 < t_1 < \cdots < t_N < t_{N+1} = T$ **by**

$$|\log S(t_{i+1}) - \log S(t_i)| = \eta, \ \ i = 0, 1, \ldots$$

$$|\log S(t_{i+1}) - \log S(t_i)| < \eta \ \ \textbf{for} \ \ t_i < t < t_{i+1}$$

$$|\log S(t) - \log S(t_N)| < \eta \ \ \textbf{for} \ \ t_N < t < T.$$

- **Induced** η**-step game**

Protocol: $\mathcal{K}_0 = 1$.

$\mathcal{K}_n = \mathcal{K}_{n-1}(1 + \alpha_n x_n), \ n = 1, \ldots, N$,

where $x_n = e^{\eta} - 1$ **or** $e^{-\eta} - 1$.

- **The optimal strategy for Skeptic is** $\alpha^* = (e^\eta - 1)/(e^\eta + 1)$ **and for Reality** $p^* = 1/(e^\eta + 1)$**.**

Theorem 6 *Optimal strategy for Reality for all induced* η*-step game is:*

$S(t)$ *is a positive martingale.*

This implies that there exists a path-dependent and future-independent time change $t = t(\tau)$ *such that* $\log S^*(\tau) = \log S(t(\tau))$ *is a Brownian motion with a drift.*

Upper price of a derivative $\phi(S(T))$ depending on $S(T)$.

- **In the η-step game**
  - **For fixed N** $\quad (q^* = 1 - p^*)$

$$E_{p^*}(\phi(S(T))) \sim \int \frac{1}{\sqrt{2\pi N p^* q^*}} \phi(e^u) \exp(-\frac{u^2}{2Np^*q^*}) du$$

  - **Since N depends on the path,**

$$E_{p^*}(\phi(S(T))) \sim \sup E_N \Big(\int \frac{1}{\sqrt{2\pi N p^* q^*}} \phi(e^u) \exp(-\frac{u^2}{2Np^*q^*}) du \Big),$$

where $\sup E_N$ means supremum for all possible distribution of N.

- **Upper price of ϕ in the original game:**

$$\sup E_{\sigma^2}\Big(\int \frac{1}{\sqrt{2\pi\sigma^2}}\phi(e^u)\exp(-\frac{u^2}{2\sigma^2})du\Big),$$

where supremum is calculated for all possible distribution of $\mathrm{Var}(\log(S(T)/S(0)))$.

4.3 Strategies in induced η-step games

1) One stage strategy

- **If Skeptic assumes that Reality is stochastic and her distribution is i.i.d.** $p = \Pr(x_n = e^\eta - 1)$**, then Skeptic's best strategy is**

$$\alpha^* = \frac{pe^\eta - q}{e^\eta - 1} \qquad (q = 1 - p).$$

- **Under this Skeptic's strategy the capital process is**

$$\log \mathcal{K}_n = m_n \log \frac{p}{p^*} + (n - m_n) \log \frac{q}{q^*},$$

where $m_n = \#\{h \mid x_h = e^\eta - 1, 1 \le h \le n\}$ **is the number of heads. This formula is path-wise true.**

- **When Skeptic does not want to specify** p**, he can use the Bayesian strategy**

$$\hat{\alpha}_n^* = \frac{\hat{p}_{n,c} e^\eta - \hat{q}_{n,c}}{e^\eta - 1}, \quad \hat{p}_{n,c} = \frac{m_n + c}{n + 2c}, \ c > 0.$$

Proposition 9 *Let $\hat{p}_N = m_N/N$. Under the Bayesian strategy, Skeptic's capital is written as*

$$\log \mathcal{K}_N = N D(\hat{p}_N \mid p^*) - \frac{1}{2}\log N + O(1).$$

If $\hat{p}_N$ is close to p^*, then $\log \mathcal{K}_N$ is further approximated as

$$\log \mathcal{K}_N = \frac{N}{2}\frac{(\hat{p}_N - p^*)^2}{\hat{p}_N \hat{q}_N} - \frac{1}{2}\log N + O(1).$$

- N **depends on the path** $S(t), 0 \le t \le T$, **and** η.
- **We consider letting** $\eta \downarrow 0$. **Write** N_η.

- $$\begin{aligned} \eta(2m_{N_\eta} - N_\eta) &= \log S(T) - \log S(0) = L, \\ \log \mathcal{K}_N &\sim \frac{N_\eta}{8}\Big(\eta + \frac{L}{\eta N_\eta}\Big)^2 - \frac{1}{2}\log N_\eta + O(1) \\ &= \frac{\eta^2 N_\eta}{8} + \frac{L^2}{8\eta^2 N_\eta} - \frac{1}{2}\log N_\eta + O(1). \end{aligned}$$

- **It follow that if $N_\eta = O(\eta^{-2-\epsilon})$ or if $L \neq 0$ and $N_\eta = O(\eta^{-2+\epsilon})$ for some $\epsilon > 0$, then $\mathcal{K}_{N_\eta} \to \infty$ as $\eta \to 0$.**

- **This implies $\sqrt{dt}$ effect, i.e., the variation exponent of $S(t)$ is 2.**

1) Two stage Markov strategy

- **Consider the sign pattern of Realities moves** x_n**, e.g.**

$$+ + - + - - + - \cdots$$

- **Let** $\hat{\alpha}_n^*$ **depend on the sign of** x_{n-1}**.**

$$\hat{\alpha}_n^* = \begin{cases} \hat{\alpha}_{n+}^* & \textbf{if } x_{n-1} > 0, \\ \hat{\alpha}_{n-}^* & \textbf{if } x_{n-1} < 0, \end{cases}$$

where

$$\hat{\alpha}_{n\pm}^* = \frac{\hat{p}_{n,c}^{\pm} e^{\eta} - \hat{q}_{n,c}^{\pm}}{e^{\eta} - 1}.$$

- **Let**

$$n_1 = \#(++),\ n_2 = \#(+-),\ n_3 = \#(-+),\ n_4 = \#(--).$$

- **Then by counting hitting times of grids with the grid size of η within the grids of size 2η, we have the following relations.**

$$n_1 + n_4 = N_{2\eta} \pm 1, \qquad 2(n_1 + n_2 + n_3 + n_4) = N_\eta,$$
$$2\eta(n_1 - n_2) = L, \qquad n_2 = n_3 \pm 1,$$
$$n_1 = \frac{1}{2}(N_{2\eta} + \frac{L}{2\eta}), \qquad n_4 = \frac{1}{2}(N_{2\eta} - \frac{L}{2\eta}),$$
$$n_2 = n_3 = \frac{1}{2}(\frac{N_\eta}{2} - N_{2\eta}).$$

Proposition 10 *Under the two stage Markov strategy the capital process is written as*

$$\log \mathcal{K}_{N_\eta} = (n_1+n_2)D(\hat{p}_+ \mid p^*) + (n_3+n_4)D(\hat{p}_- \mid p^*)$$
$$- \frac{1}{2}\log(n_1+n_2) - \frac{1}{2}\log(n_3+n_4) + O(1),$$

where

$$\hat{p}_+ = \frac{n_1}{n_1+n_2}, \quad \hat{p}_- = \frac{n_3}{n_3+n_4}.$$

- **As** $\eta \downarrow 0$**,**

$$\log \mathcal{K}_{N_\eta} = \frac{1}{8}(N_\eta + \frac{L}{\eta})(\hat{p}_+ - p^*)^2 + \frac{1}{8}(N_\eta - \frac{L}{\eta})(\hat{p}_- - p^*)^2 - \frac{1}{2}\log(N_\eta^2 - \frac{L^2}{\eta^2}) + O(1).$$

- $\log \mathcal{K}_{N_\eta} \to \infty$ **as** $\eta \to 0$ **if for some** $\epsilon > 0$

either $N_\eta \eta^{2+\epsilon} \to \infty$ **or** $N_\eta \eta^{2-\epsilon} \to 0.$

- **With Markov strategy, we can also describe how fast** $\mathcal{K}_{N_\eta}$ **diverges when the variation exponent of** $S(t)$ **deviates from 2.**