Exponential inequalities and the law of the iterated logarithm in unbounded forecasting games

Shin-ichiro Takazawa
Department of Mathematics, Graduate School of Science, Kobe University

THE UNBOUNDED FORECASTING GAME WITH A SINGLE HEDGE

Protocol:

$\mathcal{K}_0 := 1.$

FOR $n = 1, 2, \dots :$

Forecaster announces $v_n > 0.$

Skeptic announces $M_n \in \mathbb{R}$ **and** $V_n \geq 0.$

Reality announces $x_n \in \mathbb{R}.$

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(h(x_n) - v_n).$

END FOR

$$S_n = \sum_{i=1}^{n} x_i, \;\; A_n = \sum_{i=1}^{n} v_i.$$

The Unbounded Forecasting Game with the Hedge $h(x) = x^2$

Protocol:

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \ldots$:

Forecaster announces $v_n > 0$.

Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(x_n^2 - v_n)$.

END FOR

- In Chapter 5 of Shafer and Vovk (2001), a game-theoretic version of the law of the iterated logarithm (LIL) is proved under the hedge $h(x) = x^2$.

Theorem 1 (Theorem 5.2 of Shafer and Vovk (2001)). *Let* $S_n = \sum_{i=1}^n x_i$ *and* $A_n = \sum_{i=1}^n v_i$. *Then, in the unbounded forecasting game with the quadratic hedge* $h(x) = x^2$, *Skeptic can force*

$$A_n \to \infty \text{ and } |x_n| = o\left(\sqrt{\frac{A_n}{\log\log A_n}}\right)$$

$$\Rightarrow \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2A_n \log\log A_n}} \leq 1.$$

- **In the viewpoint of game-theoretic probability, Reality's moves and Forecaster's moves can not necessarily be limited.**

$$\Downarrow$$

We want to drop conditions on Reality's moves x_n.

- **Find $h(x)$ s.t.**

  the LIL holds.

  no condition on x_n.

Namely...

- **Investigate hedges $h(x)$ s.t. for some constant $c > 0$, Skeptic can force**

$$A_n \to \infty \ \Rightarrow \ \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2cA_n \log\log cA_n}} \leq 1.$$

THE UNBOUNDED FORECASTING GAME WITH THE HEDGE

$h(x) = \exp(x^2/2) - 1$

Protocol:

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \dots$:

Forecaster announces $v_n > 0$.

Skeptic announces $M_n \in \mathbb{R}$ **and** $V_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(\exp(x_n^2/2) - 1 - v_n)$.

END FOR

$$S_n = \sum_{i=1}^{n} x_i, \; C_n = 2\sum_{i=1}^{n} v_i (= 2A_n).$$

Theorem 2 (Takazawa (2009)). *Let* $\boldsymbol{S_n = \sum_{i=1}^n x_i}$ *and* $\boldsymbol{A_n = \sum_{i=1}^n v_i}$. *In the unbounded forecasting game with the hedge* $\boldsymbol{h(x) = \exp(x^2/2) - 1}$, *if Skeptic uses the following strategy* $\boldsymbol{\mathcal{P}}$:

$$\boldsymbol{M_n = t \exp\left(-\frac{t^2}{1-|t|} v_n\right) \mathcal{K}_{n-1},}$$
$$\boldsymbol{V_n = \frac{1}{v_n}\left\{1 - \exp\left(-\frac{t^2}{1-|t|} v_n\right)\right\} \mathcal{K}_{n-1}.}$$

then Skeptic's capital process is bounded as

$$\boldsymbol{\mathcal{K}_n^{\mathcal{P}} \geq \exp\left(tS_n - \frac{t^2}{1-|t|} A_n\right),}$$

for any $\boldsymbol{t \in (-1, 1)}$.

- **In Theorem 2, we consider the family of the strategy M_n and V_n depending on the parameter $t \in (-1, 1)$.**
- **We can obtain the LIL and exponential inequalities by choosing the parameter t well.**

Theorem 3 (T. (2009))**.** *Let* $S_n = \sum_{i=1}^n x_i$ *and* $C_n = 2\sum_{i=1}^n v_i$. *Then, in the unbounded forecasting game with the hedge* $h(x) = \exp(x^2/2) - 1$, *Skeptic can force*

$$\lim_{n\to\infty} C_n < \infty \;\Rightarrow\; \limsup_{n\to\infty} |S_n| < \infty,$$

$$\lim_{n\to\infty} C_n = \infty \;\Rightarrow\; \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2C_n \log\log C_n}} \leq 1.$$

Theorem 4 (T. (2009)). *Let* $S_n = \sum_{i=1}^{n} x_i$ *and* $A_n = \sum_{i=1}^{n} v_i$. *In the unbounded forecasting game with the hedge* $h(x) = \exp(x^2/2) - 1$, *for all* $a, b > 0$,

$$\bar{P}(|S_n| \geq a,\ A_n \leq b) \leq 2\exp\left\{-\left(\sqrt{a+b} - \sqrt{b}\right)^2\right\}.$$

In particular, if all v_n *are given in advance, then for all* $a > 0$,

$$\bar{P}(|S_n| \geq a) \leq 2\exp\left\{-\left(\sqrt{a+A_n} - \sqrt{A_n}\right)^2\right\}.$$

- **In the second part of Theorem 4**

  We consider the strategy s.t. t **depends on** A_n**.**

  $\Rightarrow$ **We assume that all** v_n **are given in advance.**

- **In Theorem 3 and the first part of Theorem 4**

  We consider the strategy s.t. t **does not depend on** A_n**.**

  $\Rightarrow$ **We should not assume that all** v_n **are given in advance.**

- **If we translate Theorem 2-4 in measure-theoretic terms, we have the following results.**

Theorem 5 (T. (2009)). *Let* (X_i) *be a sequence of martingale differences. Set* $S_n = \sum_{i=1}^n X_i$ *and*

$$A_n = \sum_{i=1}^{n} E\left[\exp\left(\frac{X_i^2}{2}\right) - 1 \mid \mathcal{F}_{i-1}\right].$$

For all $t \in (-1, 1)$, *define*

$$V_n(t) = \exp\left(tS_n - \frac{t^2}{1-|t|}A_n\right).$$

Then $V_n(t)$ *is a positive supermartingale with* $E[V_n(t)] \leq 1$.

- **By Theorem 5, we have the LIL and an exponential inequality for a sequence of martingale differences.**

Theorem 6 (T. (2009)). *Let* (X_i) *be a sequence of martingale differences. Set* $S_n = \sum_{i=1}^{n} X_i$ *and*

$$C_n = 2\sum_{i=1}^{n} E\left[\exp\left(\frac{X_i^2}{2}\right) - 1 \mid \mathcal{F}_{i-1}\right] = (2A_n).$$

If $C_n < \infty$ *a.s. for all* n *and* $\lim_{n\to\infty} C_n = \infty$ *a.s., then*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2C_n \log\log C_n}} \leq 1 \quad a.s.$$

Theorem 7 (T. (2009)). *Let* (X_i) *be a sequence of martingale differences. Set* $S_n = \sum_{i=1}^n X_i$ *and* $A_n = \sum_{i=1}^n E\left[\exp\left(X_i^2/2\right) - 1 \mid \mathcal{F}_{i-1}\right]$. *Then, for all* $a, b > 0$,

$$P(|S_n| \geq a,\ A_n \leq b) \leq 2\exp\left\{-\left(\sqrt{a+b} - \sqrt{b}\right)^2\right\}.$$

- **Theorem 7 is an extension of Azuma-Hoeffding's inequality.**

Theorem 8 (Azuma-Hoeffding's inequality, Hoeffding (1963), Azuma (1967)). *If* (X_i) *is a sequence of martingale differences such that* $|X_i| \leq c_i$ *for* $c_i \geq 0$*, then for all* $a \geq 0$*,*

$$P(|S_n| \geq a) \leq 2\exp\left(-\frac{a^2}{2\sum_{i=1}^n c_i^2}\right),$$

where $S_n = \sum_{i=1}^n X_i$*.*

- **Azuma-Hoeffding's inequality essentially depends on the boundedness for martingale differences.**
- **Theorem 7 does not assume the boundedness for martingale differences.**

THE UNBOUNDED FORECASTING GAME WITH THE HEDGE $h(x) = e^{|x|} - 1$

Protocol:

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \dots$:

Forecaster announces $v_n > 0$.

Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(e^{|x_n|} - 1 - v_n)$.

END FOR

Skeptic can force

$$\lim_{n\to\infty} B_n = \infty \;\Rightarrow\; \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2B_n \log\log B_n}} \leq 1,$$

where

$$S_n = \sum_{i=1}^{n} x_i,\; B_n = 2e^{-1} \sum_{i=1}^{n} v_i.$$

The Unbounded Forecasting Game with the Hedge $h(x) = x^2$

Protocol:

$\mathcal{K}_0 := 1.$

FOR $n = 1, 2, \ldots:$

Forecaster announces $v_n > 0$.

Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(x_n^2 - v_n).$

END FOR

- **We derive an exponential inequality in the unbounded forecasting game with the hedge $h(x) = x^2$.**

$$S_n = \sum_{i=1}^{n} x_i,\ T_n = \sum_{i=1}^{n} x_i^2,\ A_n = \sum_{i=1}^{n} v_i.$$

Theorem 9 (T. (2009))**.** *Let* $\boldsymbol{S_n = \sum_{i=1}^n x_i}$ *and* $\boldsymbol{T_n = \sum_{i=1}^n x_i^2}$*. Set* $\boldsymbol{A_n = \sum_{i=1}^n v_i}$*. In the unbounded forecasting game with the hedge* $\boldsymbol{h(x) = x^2}$*, if Skeptic uses the following strategy* $\boldsymbol{\mathcal{P}}$*:*

$$\boldsymbol{M_n = t \exp\left(-\frac{t^2}{2} v_n\right) \mathcal{K}_{n-1},}$$
$$\boldsymbol{V_n = \frac{1}{v_n}\left\{1 - \exp\left(-\frac{t^2}{2} v_n\right)\right\} \mathcal{K}_{n-1}.}$$

then Skeptic's capital process is bounded as

$$\boldsymbol{\mathcal{K}_n^{\mathcal{P}} \geq \exp\left(t S_n - \frac{t^2}{2}(T_n + A_n)\right),}$$

for any $\boldsymbol{t} \in \mathbb{R}$*.*

Theorem 10 (T. (2009))**.** *Let* $\boldsymbol{S_n = \sum_{i=1}^n x_i}$ *and* $\boldsymbol{T_n = \sum_{i=1}^n x_i^2}$*. Set* $\boldsymbol{A_n = \sum_{i=1}^n v_i}$*. In the unbounded forecasting game with the hedge* $\boldsymbol{h(x) = x^2}$*, for all* $\boldsymbol{a, b > 0}$*,*

$$\bar{\boldsymbol{P}}\left(|\boldsymbol{S}_n| \geq \boldsymbol{a},\ \boldsymbol{T}_n + \boldsymbol{A}_n \leq \boldsymbol{b}\right) \leq \mathbf{2}\exp\left(-\frac{\boldsymbol{a}^2}{\mathbf{2}\boldsymbol{b}}\right).$$

- **A corresponding measure-theoretic result to Theorem 9-10 is derived by Bercu and Touati (2008).**

Theorem 11 (Bercu and Touati (2008) Ann. Appl. Probab., 18, 1848–1869). *Let* (X_i) *be a sequence of martingale differences with* $E[X_i^2] < \infty$. *Set* $S_n = \sum_{i=1}^n X_i$ *and* $T_n = \sum_{i=1}^n X_i^2$. *Put* $A_n = \sum_{i=1}^n E[X_i^2 \mid \mathcal{F}_{i-1}]$. *For any* $t \in \mathbb{R}$ *and for all* $n \geq 0$, *define*

$$V_n(t) = \exp\left(tS_n - \frac{t^2}{2}(T_n + A_n)\right)$$

Then, $V_n(t)$ *is a positive supermartingale with* $E[V_n(t)] \leq 1$. *Furthermore, for any* $a, b > 0$,

$$P(|S_n| \geq a,\ T_n + A_n \leq b) \leq 2\exp\left(-\frac{a^2}{2b}\right).$$

- **Bercu-Touati's inequality is an extension of Azuma-Hoeffding's inequality.**

- **By Theorem 9, we have the following result.**

Theorem 12 (T. (2010)). *Let* $S_n = \sum_{i=1}^n x_i$ *and* $T_n = \sum_{i=1}^n x_i^2$. *Set* $A_n = \sum_{i=1}^n v_i$. *Then, in the unbounded forecasting game with the hedge* $h(x) = x^2$, *Skeptic can force*

$$\lim_{n\to\infty} A_n = \infty \ \Rightarrow\ \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2(T_n + A_n)\log\log(T_n + A_n)}} \le 1.$$

- **Investigate the protocol s.t.**

  Skeptic can force $T_n = O(A_n)$**.**

  the hedge is not a exponential function but a polynomial.

$$\Downarrow$$

Consider the unbounded forecasting game with double hedges.

The Unbounded Forecasting Game with Double Hedges

Protocol:

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \dots$:

Skeptic announces $M_n \in \mathbb{R}$, $V_n \geq 0$ **and** $W_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(x_n^2 - v) + W_n(x_n^4 - w)$,

$v, w > 0$.

END FOR

Theorem 13 (T. (2010)). *Let* $S_n = \sum_{i=1}^n x_i$. *In the unbounded forecasting game with double hedges, Skeptic can force*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n \log\log n}} \leq 2\sqrt{v}.$$