Comments on Professor Takeuchi's talk

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Third Workshop on Game-Theoretic Probability and Related Topics

Royal Holloway, University of London

- **Game-theoretic probability and von Neumann’s minimax theorem**

- Jean André Ville’s proof of the minimax theorem

- Jean André Ville and Abraham Wald

Two-person, zero-sum game with finite strategies.

Players I and II

Von Neumann's minimax theorem:

There exists a mixed strategy for each player and a number $\boldsymbol{V}$ such that

- Given II's strategy,
  (best payoff possible for I) = $\boldsymbol{V}$.

- Given I's strategy,
  (best payoff possible for II) = $-\boldsymbol{V}$.

Game-theoretic probability starts with three players.

The one-round game ("unit game"):

> **Forecaster** offers bets.
> **Skeptic** chooses a bet.
> **Reality** decides the outcome.

How to make it two-person zero sum:

- Treat it as a game between Skeptic and Forecaster-Reality.
- Fix a function of the player's moves and treat it as Skeptic's gain and Forecaster-Reality's loss.
- The function is not, in general, Skeptic's capital.

Forecaster/Reality offers bets.
Skeptic chooses a bet.
Forecaster/Reality decides the outcome.

Fix a function of the players' moves.
Treat it as Skeptic's gain and Forecaster/Reality's loss.

Each theorem in Shafer/Vovk 2001 uses a different function.

Do we allow mixed strategies?

- **The move space for Skeptic is generally a convex cone. So Skeptic can mix his strategies.**
- **For Forecaster/Reality, the role of a mixed strategy is not so clear.**

FOR $n = 1, 2, \ldots, N$:

Forecaster offers a cone of bets $\mathcal{C}_n$.

Skeptic chooses $f_n \in \mathcal{C}_n$.

Reality announces $y_n \in \mathcal{Y}$.

Skeptic's capital changes by $f_n(y_n)$.

Suppose $\mathcal{Y}$ is finite.

$\mathcal{C}_n$ is a set of functions on $\mathcal{Y}$.

In other words, $\mathcal{C}_n$ is a convex cone in Euclidean space.

On a single round,

$$\overline{\mathbb{E}}_{\mathcal{C}}(g) := \inf_{f \in \mathcal{C}} \max_{y \in \mathcal{Y}} \left(g(y) - f(y)\right)$$

Professor Takeuchi connects the probability game with the minimax theorem in a different way.

The unit zero-sum two-person game:

- Pay-off matrix $A = \{a_{ij}\} : k \times m$.
- The first player: Skeptic
- The second player: Reality
- There are k items Skeptic can bet and m outcomes Reality can choose.
- The value of the game $v^* = v^*(A)$.

Skeptic is the row player. If I understand correctly, Professor Takeuchi assumes that the pay-off matrix has the special property that the value of the game is zero, as in matching pennies.

So if we allow Skeptic to scale his bets (not just mix them), Skeptic can obtain the probability game's' value of zero in a different way than by making zero moves.

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What are some interesting examples of probability games with such pay-off matrices?

- Game-theoretic probability and von Neumann’s minimax theorem

- **Jean André Ville’s proof of the minimax theorem**

- Jean André Ville and Abraham Wald

Jean André Ville

Born 1910.

1929-1932: First normalian to study probability with Borel and Fréchet.

1933-34: Berlin to study analysis.

1934-35: Vienna to study with Menger and Wald.

1935-36: Wrote up his invention of game-theoretic probability. Fréchet would not accept it as a thesis.

1936-37: Wrote up Borel's lectures on game theory, gave convex-analysis proof of minimax theorem.

1937-38: Studied Doob with Wolfgang Doeblin, put more analysis in his thesis.

March 1939: Defended and published thesis.

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