How rough can security prices be in idealized financial markets?

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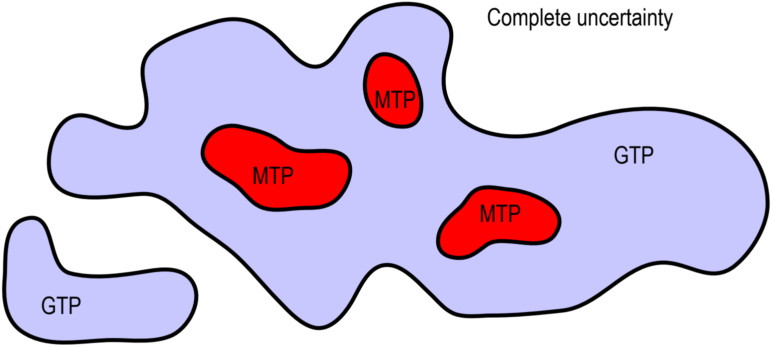
The context

This talk: further development of my 2008 talk at the previous workshop (but self-contained). That talk was inspired by the paper by Takeuchi, Kumon, Takemura.

Main difference: in this talk I will concentrate on càdlàg (continuous on the right with limits on the left) rather than continuous price paths. (The assumption of continuity is often criticized: McCullagh,. . . , whereas càdlàg processes are the mainstay of probability theory.) Connections with the work by de Rooij and Koolen (next talk).

Next slide: my picture from the previous talk.


Complete uncertainty
MTP
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Outline

1. Càdlàg price paths
   - Basic definitions
   - Brownian motion is wild enough
2. Continuous price paths
   - Definition and motivation
   - Emergence of volatility
   - Cover-type definitions
3. Right-continuous price paths and discussion

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Basic game

Players: Reality (market) and Skeptic (speculator).

Time: $[0, T]$.

Two steps of the game:

- Skeptic chooses his trading strategy.
- Reality chooses a positive (=non-negative) càdlàg function $\omega : [0, T] \to [0, \infty)$ (the price path).

Ω: all càdlàg functions $\omega : [0, T] \to [0, \infty)$.

Processes etc.

$\mathcal{F}_t^\circ$, $t \in [0, T]$: the smallest σ-algebra on Ω that makes all functions $\omega \mapsto \omega(s)$, $s \in [0, t]$, measurable; $\mathcal{F}_t$: the universal completion of $\mathcal{F}_t^\circ$.

A process S: a family of functions $S_t : \Omega \to \mathbb{R}$, $t \in [0, \infty)$, each S_t being $\mathcal{F}_t$-measurable; its paths: $t \mapsto S_t(\omega)$.

Event: any subset of Ω; not necessarily an element of $\mathcal{F} := \mathcal{F}_\infty := \vee_t \mathcal{F}_t$.

Stopping times $\tau : \Omega \to [0, T] \cup \{\infty\}$ w.r.t. to $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual.

Allowed strategies I

A simple trading strategy G consists of:

- an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ such that, for any $\omega \in \Omega$, $\tau_n(\omega) < \infty$ for only finitely many n
- for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable h_n

To such G and initial capital $c \in \mathbb{R}$ corresponds the simple capital process

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr)$$

(interpretation: the interest rate is zero).
$h_n(\omega)$: Skeptic's bet (or stake) at time τ_n
$\mathcal{K}_t^{G,c}(\omega)$: Skeptic's capital at time t

Allowed strategies II

A positive capital process is a process S with values in $[0, \infty]$ that can be represented as

$$S_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n, c_n}(\omega),$$

where

- the simple capital processes $\mathcal{K}_t^{G_n, c_n}(\omega)$ are required to be positive, for all t and ω
- the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge

Null events

$E \subseteq \Omega$ is said to be null if there exists a positive capital process S with $S_0 = 1$ such that $S_T(\omega) = \infty$ for all $\omega \in E$.

Intuition: you can become infinitely rich risking only £1 if the event is singled out in advance and happens.

Phrases applied to the complement of a null event: almost certain, almost surely (a.s.), for almost all ω.

Variation

For each $p \in (0, \infty)$, the p-variation of $\omega \in \Omega$ is

$$\mathrm{v}_p(\omega) := \sup_{\kappa} \sum_{i=1}^{n} |\omega(t_i) - \omega(t_{i-1})|^p,$$

where n ranges over $\mathbb{N}$ and κ over all partitions $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$.

There exists a unique number $\mathrm{vi}(\omega) \in [0, \infty]$ (the variation index of ω) such that:

$$\mathrm{v}_p(\omega) \begin{cases} < \infty & \text{when } p > \mathrm{vi}(\omega) \\ = \infty & \text{when } p < \mathrm{vi}(\omega) \end{cases}$$

Notice: $\mathrm{vi}(\omega) \notin (0, 1)$ when ω is continuous.

Efficient markets—moderate volatility

Theorem

For almost all $\omega \in \Omega$,

$$\mathrm{vi}(\omega) \le 2.$$

Proof.

Idea: if the price path crosses an interval (a, b) too often, we can become rich by buying below a and then selling above b. Take

$$(a, b) := \left(k2^{-j}, (k+1)2^{-j}\right)$$

and combine over all $j, k = 0, 1, 2, \ldots$. □

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Assumption of continuity

This section: Ω is the set of all continuous functions $\omega : [0, T] \to \mathbb{R}$. (This time we do not have to insist that ω should be positive!)

We can define $\mathcal{F}_t$ as before, or by $\mathcal{F}_t := \mathcal{F}_t^\circ$.

The definition of null events is as before.

Discussion of the assumption

Are security price paths continuous functions “in reality”?

No: they are not functions at all.

- Bid prices are different from ask prices.
- Both bid and ask prices depend on how many units we want to buy/sell.

But in many contexts (“normal market conditions”) this assumptions is reasonable. It allows us to use “stop-loss strategies’; opens up new possibilities.

Sources of volatility

What creates volatility?

- News?
- Merely the process of trading?

Our model with continuous price paths: 2 is true.


FTSE 100; Last:5238, Hi:6731, Lo:3512
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4600
4400
4200
4000
3800
3600
3400
3200
Value
Sep 07
Jan 08
May
Sep
Jan 09
May
Sep
Jan 10
May

Emergence of volatility

Theorem

For almost all $\omega \in \Omega$,

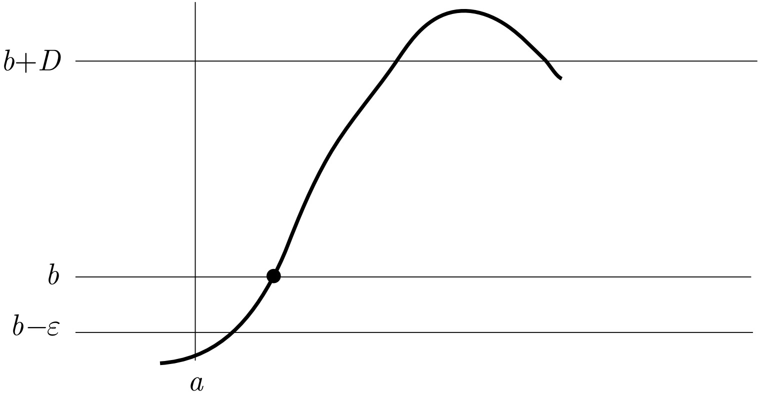
$$\mathrm{vi}(\omega) \in \{0, 2\}.$$

This is the result I mentioned in 2008, a version of Takeuchi et al. (2009) and Shafer and Vovk (2003).

Proof.

Idea: To see that we can profit from a price path ω that is too regular, see the next slide (which assumes that ω strictly increases over an interval). □

How to profit from an increasing price path



Super-prudent strategies

A simple trading strategy is prudent (for a given initial capital) if it leads to a positive capital process S. It is super-prudent if it always holds both security and cash in positive quantities. In other words: for all ω and t,

1. $h_t(\omega) \geq 0$
2. $S_t(\omega) - h_t(\omega)\omega(t) \geq 0$

where $h_t(\omega) := h_n(\omega)$ for $t \in (\tau_n, \tau_{n+1}]$.

Conditions 1 and 2 hold automatically for prudent strategies in the positive càdlàg case.

The Cover-type versions of the definitions

Considering only super-prudent simple trading strategies (as in Cover's and de Rooij and Koolen's work), we obtain another definition of null events. So for sets of positive continuous functions, we have 4 definitions of being null: (continuous or positive càdlàg market) $\times$ (imposing super-prudence or not).

Three of them are equivalent; the remaining one is the one for which $\mathrm{vi}(\omega) \in \{0, 2\}$ is guaranteed.

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The most intuitive version of the definitions

Let Ω be the set of all right-continuous functions $\omega : [0, T] \to [0, \infty)$. Define $\mathcal{F}_t$ as the family of all "cylinders": $E \in \mathcal{F}_t$ iff

$$\left(\omega \in E, \omega' \in \Omega, \omega|_{[0,t]} = \omega'|_{[0,t]}\right) \Longrightarrow \omega' \in E.$$

All other definitions: as before.

This choice of $\mathcal{F}_t$ makes all definitions (τ, $\mathcal{F}_\tau$,...) very simple and intuitive: we do not need to worry about measurability anymore. All the previous results hold under these definitions.

Theorem

Almost surely, the price path ω is càdlàg.

What's the catch?

Danger: is it possible that removing measurability turns the market into a money machine?

Suppose Ω is the set of all positive càdlàg (or right-continuous) functions. Dawid's 1985 construction: for any positive capital process S there exists a non-constant positive càdlàg ω such that $S_T(\omega) \leq S_0$. Also easy to see: you can't become infinitely rich risking only £1 if $\inf \omega > 0$ and $\mathrm{v}_1(\omega) < \infty$.

Question

Let Ω be the set of all positive càdlàg (or right-continuous) functions. Prove that the set of all ω with $\mathrm{vi}(\omega) = 2$ is not null.

Related papers

Kei Takeuchi, Masayuki Kumon, and Akimichi Takemura.
A new formulation of asset trading games in continuous time with essential forcing of variation exponent.
Bernoulli, 15:1243–1258, 2009

This paper introduced in game-theoretic probability the key technique of "high-frequency limit order strategies".

Vladimir Vovk.
Rough paths in idealized financial markets.
http://probabilityandfinance.com, Working Paper 35.

This paper contains proofs, or references to proofs, of all results stated in this talk.

Related talk

The next one.

Thank you for your attention!