Introduction

Formally, game-theoretic probability is a special case of imprecise probability.

Natural question: what kind of imprecise probability is game-theoretic probability?

First: it is usually not a probability measure: for some important events E,

$$\underline{\mathbb{P}}(E) < \overline{\mathbb{P}}(E).$$

Countable subadditivity

For any sequence of events (no measurability restrictions) $E_1, E_2, \ldots$,

$$\overline{\mathbb{P}}\left(\cup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \overline{\mathbb{P}}\left(E_n\right).$$

(Because we can combine our bets.)

Outer measure

So it is an outer measure in the sense of Carathéodory: a set function defined on all subsets of Ω that is nonnegative, monotone, countably subadditive, and satisfied $\overline{\mathbb{P}}(\emptyset) = 0$.

Remember that an event E is measurable if, for all $A \subseteq \Omega$,

$$\overline{\mathbb{P}}(A) = \overline{\mathbb{P}}(A \cap E) + \overline{\mathbb{P}}(A \cap E^c).$$

The measurable events constitute a σ-algebra; on the measurable events, $\overline{\mathbb{P}} = \underline{\mathbb{P}}$; and the restriction of $\overline{\mathbb{P}}$ to the measurable events is a probability measure.

This gives rise to a lot of interesting questions about the class of measurable sets, especially in the case of continuous time.

Choquet capacity

Technically, it is important that in many cases game-theoretic probability is a Choquet capacity:

- it is monotone;
- for any nested increasing sequence $E_1 \subseteq E_2 \subseteq \cdots$ of arbitrary subsets of Ω,

$$\overline{\mathbb{P}}\left(\cup_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \overline{\mathbb{P}}(E_i);$$

- for any nested decreasing sequence $K_1 \supseteq K_2 \supseteq \cdots$ of compact sets in Ω,

$$\overline{\mathbb{P}}\left(\cap_{i=1}^{\infty} K_i\right) = \lim_{i \to \infty} \overline{\mathbb{P}}(K_i).$$

This has been checked carefully in the prequential framework (with the outcome space $\{0, 1\}$), but we would expect this to be true in the situations when Forecaster's moves are Choquet capacities.

Not strongly additive

The class of capacities that is especially important from the point of view of imprecise probability is that of strongly subadditive capacities; for $\overline{\mathbb{P}}$ this would mean

$$\overline{\mathbb{P}}(E_1 \cup E_2) + \overline{\mathbb{P}}(E_1 \cap E_2) \le \overline{\mathbb{P}}(E_1) + \overline{\mathbb{P}}(E_2), \quad \forall E_1, E_2.$$

This is an example of sets E_1 and E_2 in the prequential protocol for which the condition of strong subadditivity is violated:

$$E_1 = \left\{ \left(0, 0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, 0, 0\right) \right\},$$
$$E_2 = \left\{ \left(0, 0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 1, 0, 0\right) \right\}.$$

For these subsets we have

$$\overline{\mathbb{P}}(E_1 \cup E_2) + \overline{\mathbb{P}}(E_1 \cap E_2) = 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \overline{\mathbb{P}}(E_1) + \overline{\mathbb{P}}(E_2).$$