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FACULTY OF ENGINEERING

Imprecise probabilities

and some aspects of their connection with GTP

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Walley's imprecise probability models

Accepting gambles

Consider an exhaustive set Ω of mutually exclusive alternatives ω, exactly one of which obtains.

Subject

is uncertain about which alternative obtains.

A gamble $f\colon \Omega \to \mathbb{R}$

is interpreted as an uncertain reward: if the alternative that obtains is ω, then the reward for Subject is $f(\omega)$.

Let $\mathscr{L}(\Omega)$ be the set of all gambles on Ω.

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Accepting gambles

Subject **accepts a gamble f**

if he accepts to engage in the following transaction, where

1. we determine which alternative ω obtains;
2. Subject receives $f(\omega)$.

We try to model Subject's uncertainty by looking at which gambles in $\mathscr{L}(\Omega)$ he accepts.

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Coherent sets of really desirable gambles

Subject specifies a set $\mathscr{R}$ of gambles he accepts, his set of really desirable gambles. $\mathscr{R}$ is called coherent if it satisfies the following rationality requirements:

D1. if $f < 0$ then $f \notin \mathscr{R}$ [avoiding partial loss];

D2. if $f > 0$ then $f \in \mathscr{R}$ [accepting partial gain];

D3. if $f_1 \in \mathscr{R}$ and $f_2 \in \mathscr{R}$ then $f_1 + f_2 \in \mathscr{R}$ [combination];

D4. if $f \in \mathscr{R}$ then $\lambda f \in \mathscr{R}$ for all non-negative real numbers λ [scaling].

Here '$f < 0$' means '$f \leq 0$ and not $f = 0$'. Walley has also argued that sets of really desirable gambles should satisfy an additional axiom:

D5. $\mathscr{R}$ is $\mathscr{B}$-conglomerable for any partition $\mathscr{B}$ of Ω: if $I_B f \in \mathscr{R}$ for all $B \in \mathscr{B}$, then also $f \in \mathscr{R}$ [full conglomerability].

Walley's imprecise probability models

Natural extension as a form of inference

Since D1–D4 are preserved under arbitrary non-empty intersections:

Theorem

Let $\mathscr{A}$ be any set of gambles. Then there is a coherent set of desirable gambles that includes $\mathscr{A}$ if and only if

$$f \not\leq 0 \text{ for all } f \in \operatorname{posi}(\mathscr{A})$$

In that case, the natural extension $\mathscr{E}(\mathscr{A})$ of $\mathscr{A}$ is the smallest such coherent set, and given by:

$$\mathscr{E}(\mathscr{A}) := \operatorname{posi}(\mathscr{A} \cup \mathscr{L}^{+}(\Omega)).$$

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Lower and upper previsions

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Given Subject's coherent set $\mathscr{R}$, we can define his upper and lower previsions:

$$\overline{P}(f) := \inf\{\alpha \colon \alpha - f \in \mathscr{R}\}$$
$$\underline{P}(f) := \sup\{\alpha \colon f - \alpha \in \mathscr{R}\}$$

so $\underline{P}(f) = -\overline{P}(-f)$.

- $\underline{P}(f)$ is the supremum price α for which Subject will buy the gamble f, i.e., accept the gamble $f - \alpha$.
- the lower probability $\underline{P}(A) := \underline{P}(I_A)$ is Subject's supremum rate for betting on the event A.

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Walley's imprecise probability models

Conditional lower and upper previsions

We can also define Subject's conditional lower and upper previsions: for any gamble f and any non-empty subset B of Ω, with indicator I_B:

$$\overline{P}(f|B) := \inf\{\alpha\colon I_B(\alpha - f) \in \mathscr{R}\}$$
$$\underline{P}(f|B) := \sup\{\alpha\colon I_B(f - \alpha) \in \mathscr{R}\}$$

so $\underline{P}(f|B) = -\overline{P}(-f|B)$ and $\underline{P}(f) = \underline{P}(f|\Omega)$.

- $\underline{P}(f|B)$ is the supremum price α for which Subject will buy the gamble f, i.e., accept the gamble $f - \alpha$, contingent on the occurrence of B.
- For any partition $\mathscr{B}$, define the gamble $\underline{P}(f|\mathscr{B})$ as

$$\underline{P}(f|\mathscr{B})(\omega) := \underline{P}(f|B), \quad B \in \mathscr{B}, \omega \in B$$

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Coherence of conditional lower and upper previsions

Suppose you have a number of functionals

$$\underline{P}(\cdot|\mathscr{B}_1), \ldots, \underline{P}(\cdot|\mathscr{B}_n)$$

These are called coherent if there is some coherent set of desirable gambles $\mathscr{R}$ that is $\mathscr{B}_k$-conglomerable, such that

$$P(f|B_k) = \sup\left\{\alpha \in \mathbb{R} \colon I_{B_k}(f-\alpha) \in \mathscr{R}\right\} \quad B_k \in \mathscr{B}_k, k = 1, \ldots, n$$

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Properties of conditional lower and upper previsions

Theorem ([10])

Consider a coherent set of really desirable gambles, let B be any non-empty subset of Ω, and let f, f_1 and f_2 be gambles on Ω. Then:

1. $\inf_{\omega\in B} f(\omega) \leq \underline{P}(f|B) \leq \overline{P}(f|B) \leq \sup_{\omega\in B} f(\omega)$ *[positivity];*
2. $\underline{P}(f_1+f_2|B) \geq \underline{P}(f_1|B)+\underline{P}(f_2|B)$ *[super-additivity];*
3. $\underline{P}(\lambda f|B) = \lambda \underline{P}(f|B)$ *for all real* $\lambda \geq 0$ *[non-negative homogeneity];*
4. *if* $\mathscr{B}$ *is a partition of* Ω *that refines the partition* $\{B, B^c\}$ *and* $\mathscr{R}$ *is* $\mathscr{B}$*-conglomerable, then* $\underline{P}(f|B) \geq \underline{P}(\underline{P}(f|\mathscr{B})|B)$ *[conglomerative property].*

Walley's imprecise probability models

Conditional previsions

If $\underline{P}(f|B) = \overline{P}(f|B) =: P(f|B)$ then $P(f|B)$ is Subject's fair price or prevision for f, conditional on B.

- It is the fixed amount of utility that I am willing to exchange the uncertain reward f for, conditional on the occurrence of B.
- Related to de Finetti's fair prices [6], and to Huygens's [7]

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Corollary

1. $\inf_{\omega \in B} f(\omega) \leq P(f|B)$;
2. $P(\lambda f + \mu g|B) = \lambda P(f|B) + \mu P(g|B)$;
3. *if $\mathscr{B}$ is a partition of Ω that refines the partition $\{B, B^c\}$ and if there is $\mathscr{B}$-conglomerability, then* $P(f|B) = P(P(f|\mathscr{B})|B)$.

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Reality's event tree and move spaces

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Reality can make a number of moves, where the possible next moves may depend only on the previous moves he has made.

- We can represent Reality's moves by an event tree.
- In each non-terminal situation t, Reality has a set of possible next moves

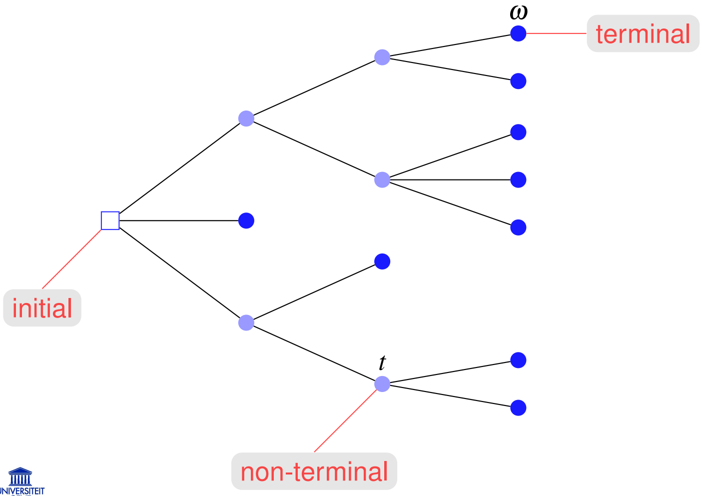
$$\mathbf{W}_t := \left\{\mathbf{w} \colon t\mathbf{w} \in \Omega^{\Diamond}\right\},$$

called Reality's move space in situation t.

- $\mathbf{W}_t$ may be infinite, but has at least two elements.
- We assume the event tree has finite horizon.

An event tree and its situations

Situations are nodes in the event tree



An event tree and its situations

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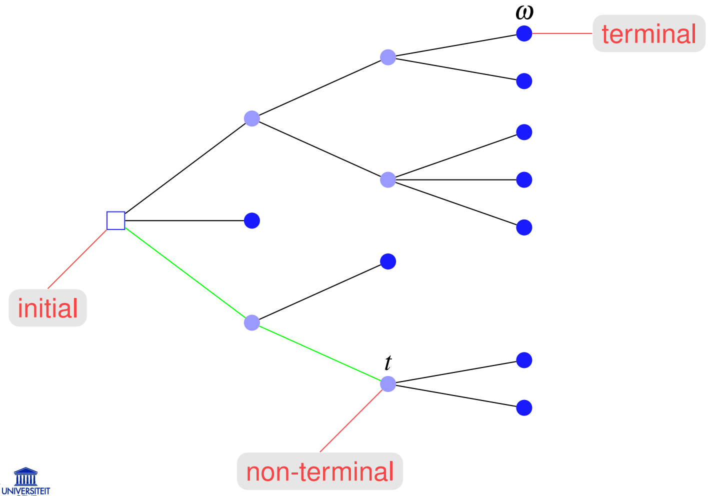

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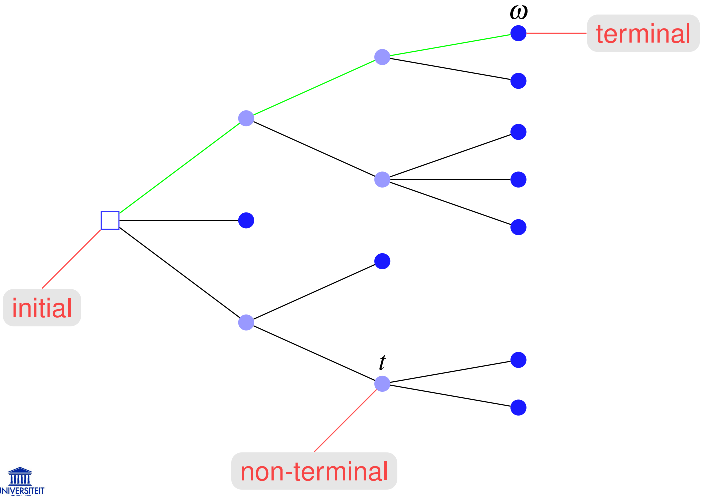

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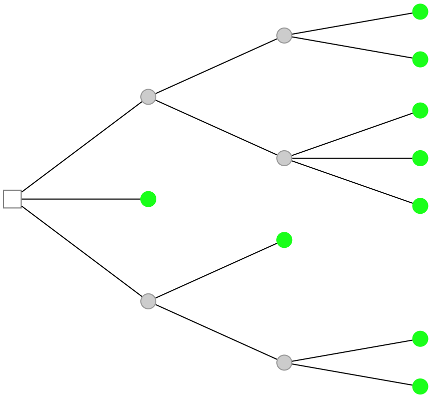
An event tree and its situations

The sample space Ω is the set of all terminal situations



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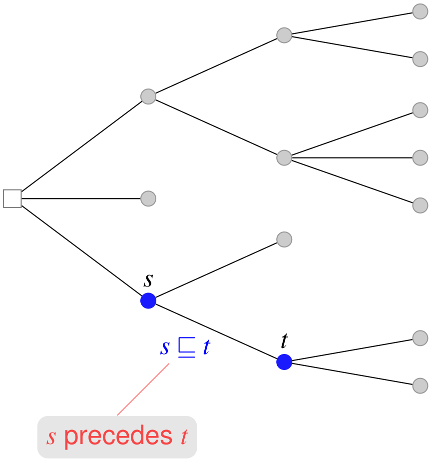
An event tree and its situations

The partial order $\sqsubseteq$ on the set $\Omega^\lozenge$ of all situations



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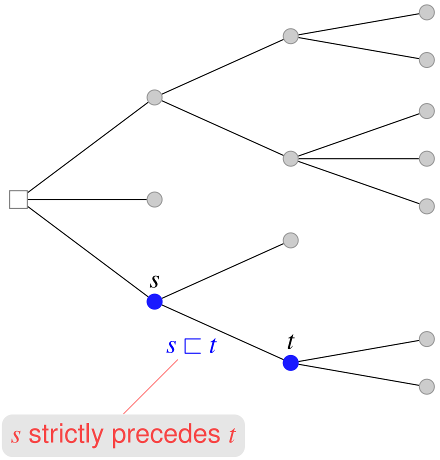
The partial order $\sqsubseteq$ on the set $\Omega^\Diamond$ of all situations



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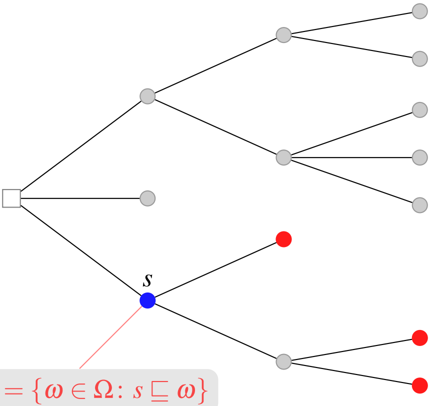
An event tree and its situations

An event A is a subset of the sample space Ω



$$E(s) = \{\omega \in \Omega \colon s \sqsubseteq \omega\}$$

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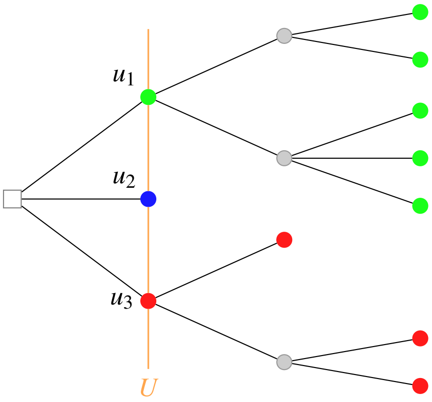
An event tree and its situations

Cuts of the initial situation



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Reality's event tree and move spaces

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Reality can make a number of moves, where the possible next moves may depend only on the previous moves he has made.

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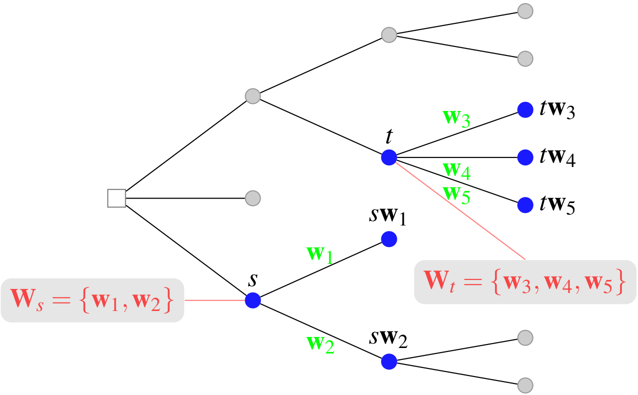
$$\mathbf{W}_t := \left\{\mathbf{w} \colon t\mathbf{w} \in \Omega^{\Diamond}\right\},$$

called Reality's move space in situation t.

- $\mathbf{W}_t$ may be infinite, but has at least two elements.
- We assume the event tree has finite horizon.

Reality's move space $\mathbf{W}_t$ in a non-terminal situation t



Coherent immediate prediction

Immediate prediction models

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In each non-terminal situation t, Forecaster has beliefs about which move $\mathbf{w}_t \in \mathbf{W}_t$ Reality will make immediately afterwards.

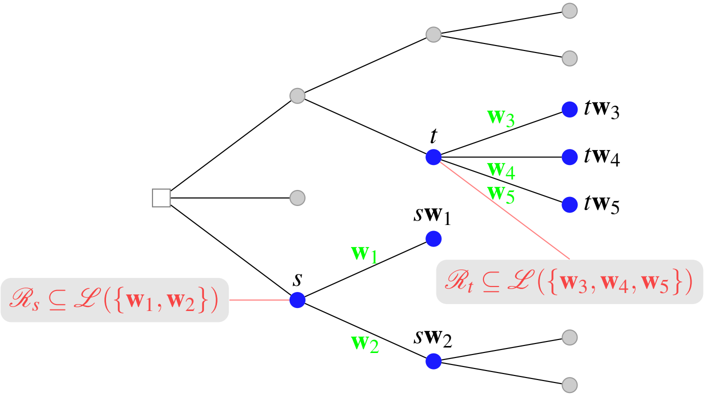
- Forecaster specifies those local predictive beliefs in the form of a coherent set of really desirable gambles $\mathscr{R}_t$ on $\mathscr{L}(\mathbf{W}_t)$.
- This leads to an immediate prediction model

$$\boxed{\mathscr{R}_t, \quad t \in \Omega^\lozenge \setminus \Omega.}$$

Coherence here means D1–D4.

Coherent immediate prediction

Immediate prediction models



Coherent immediate prediction

From a local to a global model

How to combine the local pieces of information into a global model, i.e., which gambles f on the entire sample space Ω does Forecaster accept?

- For each non-terminal situation t and each $h_t \in \mathscr{R}_t$, Forecaster accepts the gamble $I_{E(t)}h_t$ on Ω, where

$$\boxed{I_{E(t)}h_t(\omega) := \begin{cases} 0 & t \not\sqsubseteq \omega \\ h_t(\mathbf{w}) & t\mathbf{w} \sqsubseteq \omega, \mathbf{w} \in \mathbf{W}_t \end{cases}}$$

- $I_{E(t)}h_t$ represents the gamble on Ω that is called off unless Reality ends up in situation t, and then depends only on Reality's move immediately after t, and gives the same value $h_t(\mathbf{w})$ to all paths ω that go through $t\mathbf{w}$.

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Coherent immediate prediction

From a local to a global model

- So Forecaster accepts all gambles in the set

$$\boxed{\mathscr{R} := \left\{ I_{E(t)} h_t \colon h_t \in \mathscr{R}_t, t \in \Omega^\lozenge \setminus \Omega \right\}.}$$

- Find the natural extension $\mathscr{E}(\mathscr{R})$ of $\mathscr{R}$: the smallest subset of $\mathscr{L}(\Omega)$ that includes $\mathscr{R}$ and is coherent, i.e., satisfies D1–D4 and cut conglomerability.

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Cut conglomerability

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- We want predictive models, so we will condition on the $E(t)$, i.e., on the event that we get to situation t.
- The $E(t)$ are the only events that we can legitimately condition on.
- The events $E(t)$ form a partition $\mathscr{B}_U$ of the sample space Ω iff the situations t belong to a cut U.

Definition

A set of really desirable gambles $\mathscr{R}$ on Ω is cut-conglomerable (D5') if it is $\mathscr{B}_U$-conglomerable for all cuts U:

$$\boxed{(\forall u \in U)(I_{E(u)}f \in \mathscr{R}) \Rightarrow f \in \mathscr{R}.}$$

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Coherent immediate prediction

Selections and gamble processes

- A *t-selection* $\mathscr{S}$ is a process, defined on all non-terminal situations s that follow t, and such that

$$\boxed{\mathscr{S}(s) \in \mathscr{R}_s.}$$

It selects, in advance, a really desirable gamble $\mathscr{S}(s)$ from the available really desirable gambles in each non-terminal $s \sqsupseteq t$.

- With a t-selection $\mathscr{S}$, we can associate a real-valued *t-gamble process* $\mathscr{I}^{\mathscr{S}}$, which is a t-process such that for all $s \sqsupseteq t$ and $\mathbf{w} \in \mathbf{W}_s$,

$$\boxed{\mathscr{I}^{\mathscr{S}}(s\mathbf{w}) = \mathscr{I}^{\mathscr{S}}(s) + \mathscr{S}(s)(\mathbf{w}), \quad \mathscr{I}^{\mathscr{S}}(t) = 0.}$$

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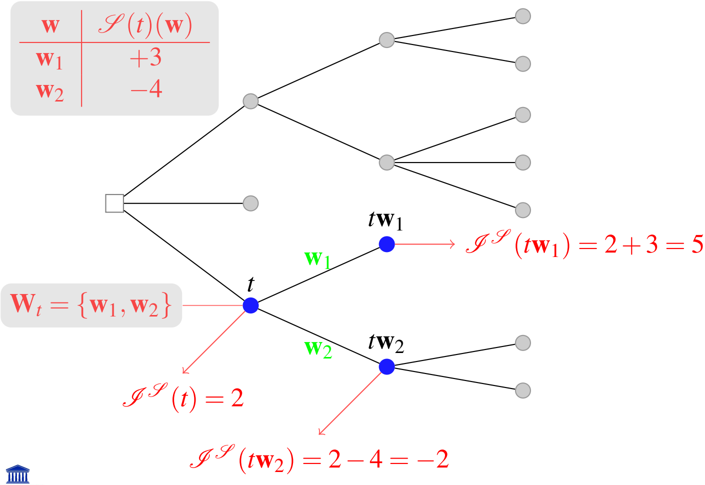
Selections and gamble processes



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Marginal Extension Theorem

Theorem (Marginal Extension Theorem)

There is a smallest set of gambles that satisfies D1–D4 and D5' and includes $\mathscr{R}$. This natural extension of $\mathscr{R}$ is given by

$$\boxed{\mathscr{E}(\mathscr{R}) := \left\{ g \in \mathscr{L}(\Omega) \colon g \geq \mathscr{I}_{\Omega}^{\mathscr{S}} \text{ for some } \square\text{-selection } \mathscr{S} \right\}.}$$

Moreover, for any non-terminal situation t and any t-gamble g, it holds that $I_{E(t)}g \in \mathscr{E}(\mathscr{R})$ if and only if there is some t-selection $\mathscr{S}_t$ such that $g \geq \mathscr{I}_{\Omega}^{\mathscr{S}_t}$.

Coherent immediate prediction

Predictive lower and upper previsions

- Use the coherent set of really desirable gambles $\mathscr{E}(\mathscr{R})$ to define special lower (and upper) previsions $\underline{P}(\cdot|t) := \underline{P}(\cdot|E(t))$ conditional on an event $E(t)$.
- For any gamble f on Ω and for any non-terminal situation t,

$$\underline{P}(f|t) := \sup\left\{\alpha \colon I_{E(t)}(f-\alpha) \in \mathscr{E}(\mathscr{R})\right\}$$
$$= \sup\left\{\alpha \colon f-\alpha \geq \mathscr{I}_{\Omega}^{\mathscr{S}} \text{ for some } t\text{-selection } \mathscr{S}\right\}.$$

- We call such conditional lower previsions predictive lower previsions for Forecaster.
- For a cut U of t, define the U-measurable t-gamble $\underline{P}(f|U)$ by $\underline{P}(f|U)(\omega) := \underline{P}(f|u)$, $u \in U$, $u \sqsubseteq \omega$.

Coherent immediate prediction

What about Skeptic's prices?

Forecaster determines Skeptic's move spaces $\mathbf{S}_t$ and gain functions λ_t as follows:

- $\mathbf{S}_t = \mathscr{R}_t$
- $\lambda_t(\mathbf{w}, h) = -h(\mathbf{w})$, where $h \in \mathscr{R}_t$ and $\mathbf{w} \in \mathbf{W}_t$
- So Skeptic can take Forecaster up on his commitments.

This leads to a coherent probability protocol, and Skeptic's lower and upper prices turn out to be identical to Forecaster's predictive lower and upper previsions.

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Properties of predictive previsions

Concatenation Theorem

Theorem (Concatenation Theorem)

Consider any two cuts U and V of a situation t such that U precedes V. Then for all t-gambles f on Ω,

1. $\underline{P}(f|t) = \underline{P}(\underline{P}(f|U)|t)$;
2. $\underline{P}(f|U) = \underline{P}(\underline{P}(f|V)|U)$.

We can calculate $\underline{P}(f|t)$ by backwards recursion, starting with $\underline{P}(f|\omega) = f(\omega)$, and using only the local models:

$$\boxed{\underline{P}_s(g) = \sup\{\alpha \colon g - \alpha \in \mathscr{R}_s\},}$$

where the non-terminal $s \sqsupseteq t$ and g is a gamble on $\mathbf{W}_s$.

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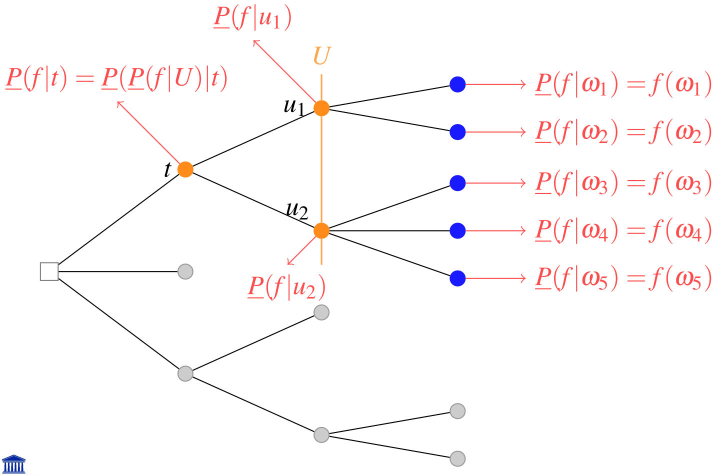

Properties of predictive previsions

Envelope theorems

- Consider in each non-terminal situation s a compatible precise model P_s on $\mathscr{L}(\mathbf{W}_s)$:

$$P_s \in \mathscr{M}_s \Leftrightarrow (\forall g \in \mathscr{L}(\mathbf{W}_s))(P_s(g) \geq \underline{P}_s(g))$$

  This leads to collection of compatible probability trees in the sense of Huygens (and Shafer).

- Use the Concatenation Theorem to find the corresponding precise predictive previsions $P(f|t)$ for each compatible probability tree.

Theorem (Lower Envelope Theorem)

For all situations t and t-gambles f, $\underline{P}(f|t)$ is the infimum (minimum) of the $P(f|t)$ over all compatible probability trees.

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Considering unbounded time

What if Reality's event tree no longer has a finite time horizon:

how to calculate the lower prices/previsions $\underline{P}(f|t)$?

The Shafer–Vovk–Ville approach

$$\sup\left\{\alpha : f-\alpha \geq \limsup \mathscr{I}^{\mathscr{S}} \text{ for some } t\text{-selection } \mathscr{S}\right\}.$$

Open question(s):

What does natural extension yield in this case, must coherence be strengthened to yield the Shafer–Vovk–Ville approach, and if so, how?

Imprecise Markov chains

Discrete-time and finite state uncertain process

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Consider an uncertain process with variables $X_1, X_2, \dots, X_n, \dots$

- Each X_k assumes values in a finite set of states $\mathscr{X}$.
- This leads to a standard event tree with situations

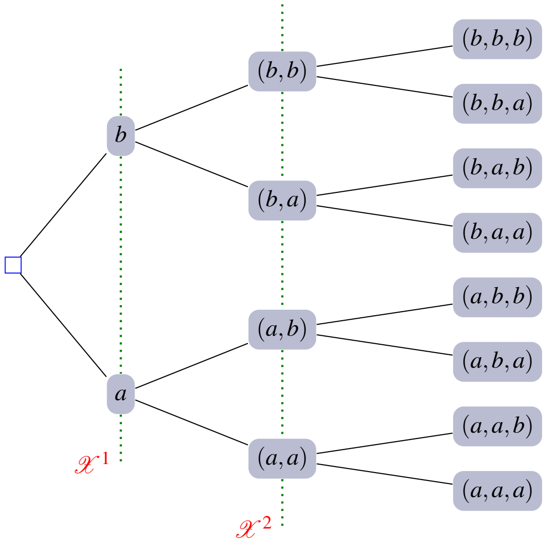
$$s = (x_1, x_2, \dots, x_n), \quad x_k \in \mathscr{X}, \quad n \geq 0$$

- In each situation s there is a local imprecise belief model $\mathscr{M}_s$: a closed convex set of probability mass functions p on $\mathscr{X}$.
- Associated local lower prevision $\underline{P}_s$:

$$\underline{P}_s(f) := \min\left\{E_p(f) \colon p \in \mathscr{M}_s\right\}; \quad E_p(f) := \sum_{x \in \mathscr{X}} f(x)p(x).$$

Imprecise Markov chains

Example of a standard event tree



Imprecise Markov chains

Precise Markov chain

The uncertain process is a (stationary) precise Markov chain when all $\mathscr{M}_s$ are singletons (precise), and

- $\mathscr{M}_\square = \{m_1\}$,
- Markov Condition:

$$\boxed{\mathscr{M}_{(x_1,\dots,x_n)} = \{q(\cdot|x_n)\}.}$$

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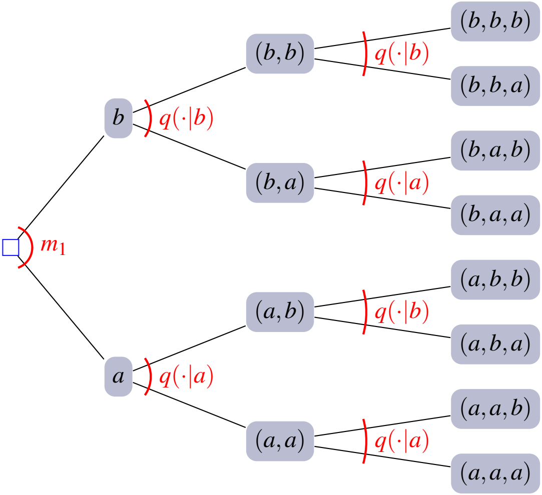
Imprecise Markov chains

Probability tree for a precise Markov chain



Imprecise Markov chains

Definition of an imprecise Markov chain

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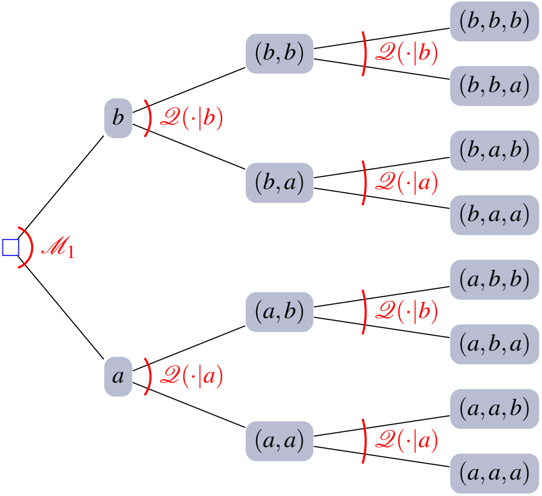
The uncertain process is a (stationary) imprecise Markov chain when the Markov Condition is satisfied:

$$\boxed{\mathscr{M}_{(x_1,\dots,x_n)} = \mathscr{Q}(\cdot|x_n).}$$

An imprecise Markov chain can be seen as an infinity of probability trees.

Imprecise Markov chains

Probability tree for an imprecise Markov chain



Imprecise Markov chains

Lower and upper transition operators

$$\underline{\mathrm{T}}\colon \mathscr{L}(\mathscr{X}) \to \mathscr{L}(\mathscr{X}) \text{ and } \overline{\mathrm{T}}\colon \mathscr{L}(\mathscr{X}) \to \mathscr{L}(\mathscr{X})$$

where for any gamble f on $\mathscr{X}$:

$$\underline{\mathrm{T}}f(x) := \min\left\{E_p(f)\colon p \in \mathscr{Q}(\cdot|x)\right\}$$
$$\overline{\mathrm{T}}f(x) := \max\left\{E_p(f)\colon p \in \mathscr{Q}(\cdot|x)\right\}$$

Then the Concatenation Formula yields:

$$\boxed{\underline{P}_n(f) = \underline{P}_1(\underline{\mathrm{T}}^{n-1}f) \text{ and } \overline{P}_n(f) = \overline{P}_1(\overline{\mathrm{T}}^{n-1}f).}$$

Complexity is linear in the number of time steps!

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Imprecise Markov chains

An example with lower and upper mass functions

$$\begin{bmatrix}\underline{\mathrm{T}}I_{\{\{a\}\}} & \underline{\mathrm{T}}I_{\{\{b\}\}} & \underline{\mathrm{T}}I_{\{\{c\}\}}\end{bmatrix} = \begin{bmatrix}\underline{q}(a|a) & \underline{q}(b|a) & \underline{q}(c|a)\\ \underline{q}(a|b) & \underline{q}(b|b) & \underline{q}(c|b)\\ \underline{q}(a|c) & \underline{q}(b|c) & \underline{q}(c|c)\end{bmatrix}$$

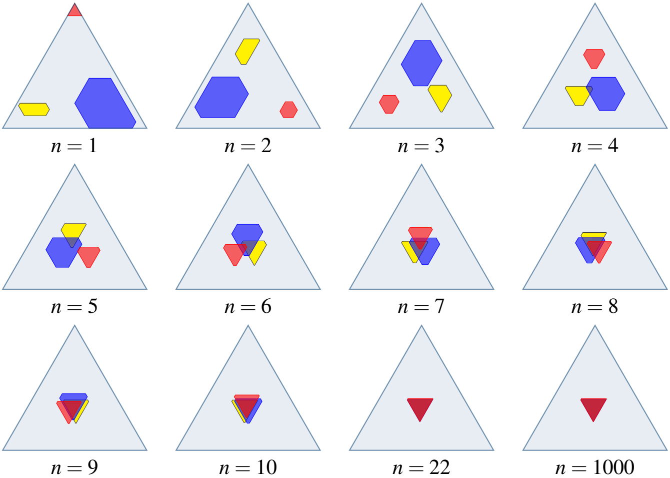
$$= \frac{1}{200}\begin{bmatrix}9 & 9 & 162\\ 144 & 18 & 18\\ 9 & 162 & 9\end{bmatrix}$$

$$\begin{bmatrix}\overline{\mathrm{T}}I_{\{\{a\}\}} & \overline{\mathrm{T}}I_{\{\{b\}\}} & \overline{\mathrm{T}}I_{\{\{c\}\}}\end{bmatrix} = \begin{bmatrix}\overline{q}(a|a) & \overline{q}(b|a) & \overline{q}(c|a)\\ \overline{q}(a|b) & \overline{q}(b|b) & \overline{q}(c|b)\\ \overline{q}(a|c) & \overline{q}(b|c) & \overline{q}(c|c)\end{bmatrix}$$

$$= \frac{1}{200}\begin{bmatrix}19 & 19 & 172\\ 154 & 28 & 28\\ 19 & 172 & 19\end{bmatrix}$$

Imprecise Markov chains

An example with lower and upper mass functions



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Imprecise Markov chains

A Perron–Frobenius Theorem

Theorem ([4])

Consider a stationary imprecise Markov chain with finite state set $\mathscr{X}$ and an upper transition operator $\overline{\mathrm{T}}$. Suppose that $\overline{\mathrm{T}}$ is regular, meaning that there is some $n > 0$ such that $\min \overline{\mathrm{T}}^n I_{\{\{x\}\}} > 0$ for all $x \in \mathscr{X}$. Then for every initial upper prevision $\overline{P}_1$, the upper prevision $\overline{P}_n = \overline{P}_1 \circ \overline{\mathrm{T}}^{n-1}$ for the state at time n converges point-wise to the same upper prevision $\overline{P}_\infty$:

$$\boxed{\lim_{n\to\infty} \overline{P}_n(h) = \lim_{n\to\infty} \overline{P}_1(\overline{\mathrm{T}}^{n-1}h) := \overline{P}_\infty(h)}$$

for all h in $\mathscr{L}(\mathscr{X})$. Moreover, the corresponding limit upper prevision $\overline{E}_\infty$ is the only $\overline{\mathrm{T}}$-invariant upper prevision on $\mathscr{L}(\mathscr{X})$, meaning that $\overline{P}_\infty = \overline{P}_\infty \circ \overline{\mathrm{T}}$.

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Independent product

Consider a number of variables X_n assuming values in a finite set $\mathscr{X}_n$, $n \in N$.

For each subset I of N, we consider the tuple X_I with components X_i, $i \in I$ assuming values in

$$\mathscr{X}_I = \times_{i \in I} \mathscr{X}_i.$$

For each variable X_n, we have a coherent marginal lower prevision:

$$\underline{P}_n \colon \mathscr{L}(\mathscr{X}_n) \to \mathbb{R}$$

Independent product

Our aim: an independent product

We want to combine these marginal lower previsions $\underline{P}_n$ into a joint lower prevision (product) for X_N

$$\underline{P}_N \colon \mathscr{L}(\mathscr{X}_N) \to \mathbb{R}$$

that models that the variables X_n, $n \in N$ are independent.

We want to extend Walley's (1991, Chapter 9) discussion of the case of two variables to the case of any finite number of variables.

What does this independence mean?

Consider any disjoint O and $I \subseteq N$.

Then X_I is epistemically irrelevant to X_O.

This irrelevance assessment allows us to infer a conditional lower prevision $\underline{P}_O(\cdot|X_I)$ from the joint $\underline{P}_N$:

$$\underline{P}_O(f|X_I) = \underline{P}_N(f) \text{ for all } f \in \mathscr{L}(\mathscr{X}_O).$$

So making the independence assessment allows us to infer from any joint lower previsions $\underline{P}_N$ a family of conditional lower previsions:

$$\mathscr{I}(\underline{P}_N) = \{\underline{P}_O(\cdot|X_I)\colon O, I \subseteq N, O \cap I = \emptyset\}.$$

Definition of an independent product

Definition

A joint lower prevision $\underline{P}_N$ is called an independent product of its marginals $\underline{P}_n$, $n \in N$ if it is coherent with the family of conditional lower previsions $\mathscr{I}(\underline{P}_N)$.

- do such independent products always exist?
- are they unique?

They are guaranteed to exist, and to be unique, for precise marginals P_n: their usual independent product

$$\times_{n\in N} P_n.$$

Definition

If it exists, then the point-wise smallest independent product of the marginals $\underline{P}_n$ is called their independent natural extension and denoted by $\otimes_{n\in N}\underline{P}_n$.

Factorising joint lower previsions

The independent product $E_N = \times_{n \in N} P_n$ of precise marginals P_n is factorising in the sense that:

$$E_N\Big(\prod_{n \in N} f_n(X_n)\Big) = \prod_{n \in N} P_n(f_n(X_n))$$

Definition

A coherent joint lower prevision $\underline{P}_N$ is called factorising if for all disjoint subsets $I, O \subseteq N$, all non-negative $f_I \in \mathscr{L}(\mathscr{X}_I)$ and all $f_O \in \mathscr{L}(\mathscr{X}_O)$:

$$\underline{P}_N(f_O f_I) = \underline{P}_N(\underline{P}_N(f_O) f_I) = \begin{cases} \underline{P}_N(f_O)\underline{P}_N(f_I) & \text{if } \underline{P}_N(f_O) \geq 0 \\ \underline{P}_N(f_O)\overline{P}_N(f_I) & \text{if } \underline{P}_N(f_O) \leq 0 \end{cases}$$

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Important theorem

Theorem

If a coherent joint lower prevision $\underline{P}_N$ is factorising, then it is an independent product of its marginals.

In other words, if $\underline{P}_N$ is factorising, then it is coherent with the family of conditional lower previsions $\mathscr{I}(\underline{P}_N)$.

Not necessarily the other way around!

Strong product

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Consider the marginals $\underline{P}_n$ and the corresponding sets of dominating linear previsions $\mathscr{M}(\underline{P}_n)$.

Consider the set of joint linear previsions:

$$\{\times_{n\in N}P_n\colon P_n\in\mathscr{M}(\underline{P}_n), n\in N\}$$

Then the strong product $\underline{S}_N=\times_{n\in N}\underline{P}_n$ of the marginals $\underline{P}_n$ is the lower envelope of this set of independent products.

Theorem

For any coherent marginal lower previsions $\underline{P}_n$, $n\in N$, their strong product $\otimes_{n\in N}\underline{P}_n$ is factorising, and therefore an independent product of these marginals.

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Consider any coherent marginal lower previsions $\underline{P}_n$, $n \in N$.

1. The independent natural extension $\otimes_{n\in N}\underline{P}_n$ exists, and is factorising.
2. For any non-empty subset R of N, $\otimes_{r\in R}\underline{P}_r$ is the $\mathscr{X}_R$-marginal for $\otimes_{n\in N}\underline{P}_s$.
3. For any partition R,S of N:

$$\otimes_{n\in N}\underline{P}_n = (\otimes_{r\in R}\underline{P}_r)\otimes(\otimes_{s\in S}\underline{P}_n).$$

$$(\otimes_{n\in N}\underline{P}_n)(f) = \sup_{\substack{h_n\in\mathscr{L}(\mathscr{X}_N)\\ n\in N}} \inf_{z_N\in\mathscr{X}_N}\left[f(z_N) - \sum_{n\in N}[h_n(z_N) - \underline{P}_n(h_n(\cdot, z_{\{n\}^c}))]\right].$$

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Credal trees

$$
\begin{tikzcd}
 & & X_1 \arrow[dl] \arrow[d] \arrow[dr] & & & \\
 & X_2 \arrow[dl] \arrow[d] & X_5 \arrow[d] & X_7 \arrow[d] \arrow[dr] & & \\
X_3 & X_4 & X_6 & X_8 & X_9 \arrow[d] \arrow[dr] & \\
 & & & & X_{10} & X_{11}
\end{tikzcd}
$$

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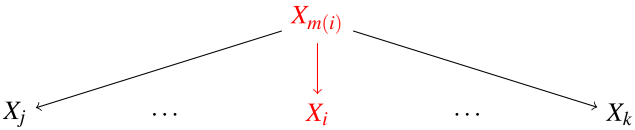
Credal trees

Local uncertainty models



- the variable X_i may assume a value in the finite set $\mathscr{X}_i$;
- for each possible value $x_{m(i)} \in \mathscr{X}_{m(i)}$ of the mother variable $X_{m(i)}$, we have a conditional lower prevision $\underline{P}_i(\cdot|x_{m(i)})\colon \mathscr{L}(\mathscr{X}_i) \to \mathbb{R}$:

$$\underline{P}_i(f|x_{m(i)}) = \text{lower prevision of } f(X_i)\text{, given that } X_{m(i)} = x_{m(i)}$$

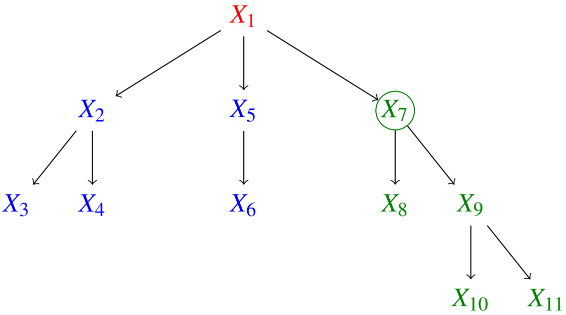
- local model $\underline{P}_i(\cdot|X_{m(i)})$ is a conditional lower prevision operator

Credal trees under epistemic irrelevance

Definition

Interpretation of graphical structure:

Conditional on the mother variable, the non-parent non-descendants of each node variable are epistemically irrelevant to it and its descendants.



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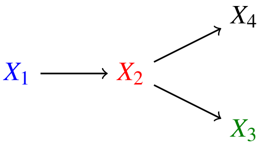
Credal trees under epistemic irrelevance

Example



- X_1 is epistemically irrelevant to X_3, conditional on X_2
- X_3 need not be epistemically irrelevant to X_1, conditional on X_2.

Conclusion

X_1 and X_3 need not be epistemically independent, conditional on X_2.

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Example

$$
\begin{tikzcd}
 & & X_4 \\
X_1 \arrow[r] & X_2 \arrow[ur] \arrow[dr] & \\
 & & X_3
\end{tikzcd}
$$

- X_3 is epistemically irrelevant to X_4, conditional on X_2
- X_4 is epistemically irrelevant to X_3, conditional on X_2.

Conclusion

X_3 and X_4 are epistemically independent, conditional on X_2.

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Credal networks under epistemic irrelevance

As an expert system

When the credal network is a (Markov) tree we can find the joint model from the local models recursively, from leaves to root.

Exact message passing algorithm

- credal tree treated as an expert system
- linear complexity in the number of nodes

Python code

- written by Filip Hermans
- testing in cooperation with Marco Zaffalon and Alessandro Antonucci

Current (toy) applications in HMMs

- character recognition [3]
- air traffic trajectory tracking and identification [1]

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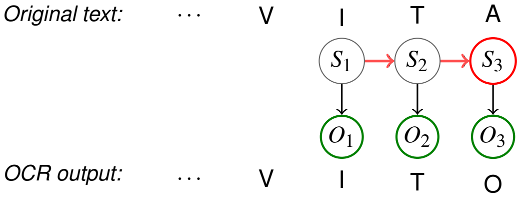
Example of application

HMMs: character recognition for Dante's Divina Commedia



Example of application

HMMs: character recognition for Dante's Divina Commedia

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Accuracy	93.96%	(7275/7743)
Accuracy (if imprecise indeterminate)	64.97%	(243/374)
Determinacy	95.17%	(7369/7743)
Set-accuracy	93.58%	(350/374)
Single accuracy	95.43%	(7032/7369)
Indeterminate output size	2.97	over 21

Table: Precise vs. imprecise HMMs. Test results obtained by twofold cross-validation on the first two chants of Dante's *Divina Commedia* and $n = 2$. Quantification is achieved by IDM with $s = 2$ and modified Perks' prior. The single-character output by the precise model is then guaranteed to be included in the set of characters the imprecise HMM identifies.

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