From Competitive Investment to Aggregating Algorithm and Defensive Forecasting

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Outline

Sequential Investment (1)

- there are M stocks $(0, 1, \ldots, M-1)$ we can invest into
  — no cash or deposit (or one of the stocks is the deposit)
  — no inflation
- time is discrete, $t = 0, 1, 2, \ldots$
- an investment decision is a vector

$$\gamma_t = (\gamma_{t,0}, \gamma_{t,1}, \ldots, \gamma_{t,M-1})$$

such that $\gamma_{t,i} \in [0, 1]$ and $\sum_{j=0}^{M-1} \gamma_{t,i} = 1$
— it shows the distribution of our capital among the stocks
— on step $t-1$ we spend the fraction $\gamma_{t,j}$ of our capital to buy stock j, $j = 0, 1, 2, \ldots, M-1$

Sequential Investment (2)

- vector x_t represents the market shift
  — let $x_{t,j}$ be the ratio of the prices of stock i at the moments t and $t-1$
  — assume $x_{t,j} \geq 0$
- between steps $t-1$ and t our capital changes from W_{t-1} to

$$W_t = \sum_{j=0}^{M-1} W_{t-1}\gamma_{t,j}x_{t,j} = W_{t-1}\langle\gamma_t, x_t\rangle$$

— if we start from $W_0 = 1$, then after T steps we get $W_T = \prod_{t=1}^{T}\langle\gamma_t, x_t\rangle$

Experts

- suppose that there are N experts $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N$ that suggest investment decisions to us
  — before deciding on γ_t, we can observe decisions $\gamma_t^{(i)}$, $i = 1, 2, \ldots, N$, output by the experts
- we want to merge experts decisions in such a way so that our capital W_t is not much less than the capital of any expert $W_t^{(i)}$, $i = 1, 2, \ldots, N$
  — we want an inequality of the type $W_t \gtrsim W_t^{(i)}$ to hold uniformly for all i and possibly for all t
- no assumptions are made as to how the experts arrive at their decisions

Laissez-Faire Merging

- we can think of a merging strategy as follows: on step $t-1$ we invest the share $p_t^{(i)}$ of our wealth as suggested by expert $\mathcal{E}_i$
- we need to decide how much money to entrust to expert $\mathcal{E}_i$
- let the experts be self-financing!
  — initially we split the capital among the experts so that $\mathcal{E}_i$ gets $p_0^{(i)}$ and then the experts operate on their share of the wealth
  — we do not redistribute the wealth
- then at every moment in time $W_t = \sum_{i=1}^N p_0^{(i)} W_t^{(i)}$ and

$$W_t \geq p_0^{(i)} W_t^{(i)}$$

Weights for the Merging

- the actual weights $p_t^{(i)}$ are given by

$$p_t^{(i)} = p_0^{(i)} W_t^{(i)} / W_t = p_0^{(i)} W_t^{(i)} / \sum_{k=1}^{N} p_0^{(k)} W_t^{(i)} \quad .$$

  — take the values $p_0^{(i)} W_t^{(i)}$ and normalise them to sum up to 1
- our investment decision is the weighted sum

$$\gamma_t = \sum_{i=1}^{N} \gamma_t^{(i)} p_t^{(i)}$$

1. Laissez-Faire Investment

2. Aggregating Algorithm

3. Defensive Forecasting

Sequential Prediction

- outcomes $\omega_t \in \Omega$ occur sequentially in discrete time: $\omega_1, \omega_2, \ldots$
- we need to output a prediction $\gamma_t \in \Gamma$ before seeing ω_t
- the quality is measured by the loss function $\lambda : \Gamma \times \Omega \to [0, +\infty]$
  — our performance is measured by the cumulative loss $\mathrm{Loss}(T) = \sum_{t=1}^{T} \lambda(\gamma_t, \omega_t)$
- a triple $\langle \Omega, \Gamma, \lambda \rangle$ (outcome space / prediction space / loss function) is called a game

Cover's Game

- let
  — $\Omega = [0, +\infty)^M$ (the positive ortant),
  — $\Gamma = \{\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_M \in \mathbb{R}^M \mid \sum_{i=0}^{M-1} \gamma_j = 1\}$ (the $(M-1)$-dimensional simplex) and
  — $\lambda(\gamma, \omega) = -\ln\langle\gamma, \omega\rangle$
- this game is called Cover's game and it describes the sequential investment scenario:

$$\mathrm{Loss}(T) = -\ln W_T$$

Binary Games

- binary games have $\Omega = \{0, 1\}$ and $\Gamma = [0, 1]$
- square-loss game has $\lambda(\gamma, \omega) = (\omega - \gamma)^2$
- absolute-loss game has $\lambda(\gamma, \omega) = |\omega - \gamma|$
- logarithmic game has

$$\lambda(\gamma, \omega) = \begin{cases} -\log_2 \gamma, & \text{if } \omega = 1; \\ -\log_2(1 - \gamma), & \text{if } \omega = 0 \end{cases}$$

- simple prediction game has

$$\lambda(\gamma, \omega) = \begin{cases} 0, & \text{if } \omega = \gamma; \\ 1, & \text{otherwise} \end{cases}$$

Prediction with Expert Advice

- suppose that we have access to predictions of N experts $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N$

  (1) FOR $t = 1, 2, \ldots$
  (2) we read $\gamma_t^{(1)}, \gamma_t^{(2)}, \ldots, \gamma_t^{(N)} \in \Gamma$
  (3) we output $\gamma_t \in \Gamma$
  (4) we observe $\omega_t \in \Omega$
  (5) END FOR

- we want to suffer loss little worse than the best expert — i.e., we want an inequality of the type $\mathrm{Loss}(t) \lesssim \mathrm{Loss}_{\mathcal{E}_i}(t)$ to hold uniformly for all i and possibly for all t

Cover's Game

- for Cover's game laissez-faire investment yields

$$W_t \geq p_0^{(i)} W_t^{(i)}$$

— by taking the logarithms we get

$$\mathrm{Loss}(t) \leq \mathrm{Loss}_{\mathcal{E}_i}(t) + \ln(1/p_0^{(i)})$$

- can we extend the algorithm to other games?

Extension of Laissez-Faire Investment (1)

- the wealth is the exponent of negative loss
  — let us introduce a parameter $\eta > 0$ (learning rate) to consider different bases
  — so $W_t^{(i)} = e^{-\eta \operatorname{Loss}_{\mathcal{E}_i}(t)}$
- consider the pseudo-algorithm that follows the laissez-faire investment strategy: it divides up its initial 'wealth' among the experts and lets them invest their shares
  — it has the notional 'wealth'

$$\widetilde{W}_t = \sum_{i=1}^{N} p_0^{(i)} W_t^{(i)}$$

Extension of Laissez-Faire Investment (2)

- but we need to produce actual predictions not 'investment decisions'
- we can calculate the weights

$$p_t^{(i)} = p_0^{(i)} W_t^{(i)} \Big/ \sum_{i=1}^{N} p_0^{(i)} W_t^{(i)}$$

$$= p_0^{(i)} e^{-\eta \operatorname{Loss}_{\mathcal{E}_i}(t)} \Big/ \sum_{k=1}^{N} p_0^{(i)} e^{-\eta \operatorname{Loss}_{\mathcal{E}_k}(t)}$$

- but we cannot take $\gamma_t = \sum_{i=1}^{N} \gamma_t^{(i)} p_t^{(i)}$ because there is no linearity in the general case

The Capital Inequality (1)

- we would like to achieve $e^{-\eta \operatorname{Loss}(t)} = W_t \geq \widetilde{W}_t$
- we can do it by making sure that
$$\frac{W_t}{W_{t-1}} \geq \frac{\widetilde{W}_t}{\widetilde{W}_{t-1}}$$
on every step
— we shall call this the capital inequality
- we have
$$\frac{W_t}{W_{t-1}} = e^{-\eta\lambda(\gamma_t,\omega_t)}$$
and
$$\frac{\widetilde{W}_t}{\widetilde{W}_{t-1}} = \frac{\sum_{i=1}^N p_t^{(i)} \widetilde{W}_{t-1} e^{-\eta\lambda(\gamma_t^{(i)},\omega_t)}}{\widetilde{W}_{t-1}} = \sum_{i=1}^N p_t^{(i)} e^{-\eta\lambda(\gamma_t^{(i)},\omega_t)}$$

The Capital Inequality (2)

- the capital inequality thus amounts to

$$e^{-\eta\lambda(\gamma_t,\omega_t)} \geq \sum_{i=1}^{N} p_t^{(i)} e^{-\eta\lambda(\gamma_t^{(i)},\omega_t)}$$

- we need to choose γ_t before we have seen ω_t so we need to make sure this holds for all ω_t
- so we need to look for γ such that

$$\lambda(\gamma,\omega) \leq -\frac{1}{\eta} \ln \sum_{i=1}^{N} p_0^{(i)} e^{-\eta\lambda(\gamma^{(i)},\omega)}$$

  for all $\omega \in \Omega$
- are there any such $\gamma \in \Gamma$?

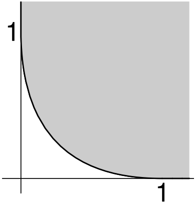
Superpredictions

- each prediction γ can be thought of as a function $g : \Omega \to [0, +\infty]$ acting as follows: $g(\omega) = \lambda(\gamma, \omega)$
- a superprediction is a function $g : \Omega \to [0, +\infty]$ such that there is $\gamma \in \Gamma$ such that $g(\omega) \geq \lambda(\gamma, \omega)$ for all $\omega \in \Omega$ — we will denote the set of superpredictions by Σ_Γ
- if Ω is finite and $\Omega = \{\omega_0, \omega_1, \ldots, \omega_{K-1}\}$, the set of superpredictions can be identified with the set of points $s = (s_0, s_1, \ldots, s_{K-1}) \in \mathbb{R}^K$ such that

$$
\begin{aligned}
s_0 &\geq \lambda(\gamma, \omega_0) \\
s_1 &\geq \lambda(\gamma, \omega_1) \\
&\ldots \\
s_{K-1} &\geq \lambda(\gamma, \omega_{K-1})
\end{aligned}
$$

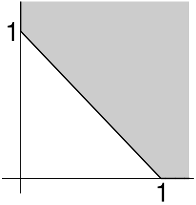
for some γ

Sets of Superpredictions for Binary Games (1)



square-loss game

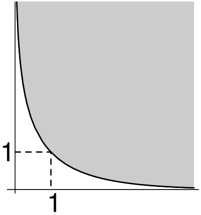
$\lambda(\gamma, \omega) = (\omega - \gamma)^2$



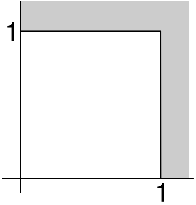
absolute-loss game

$\lambda(\gamma, \omega) = |\omega - \gamma|$

Sets of Superpredictions for Binary Games (2)



logarithmic game



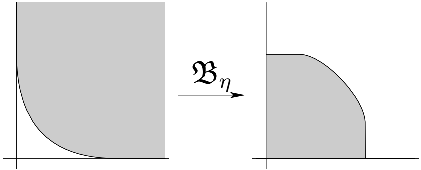
simple prediction game

$$\lambda(\gamma,\omega) = \begin{cases} -\log_2(1-\gamma), & \omega = 0 \\ -\log_2 \gamma, & \omega = 1 \end{cases}$$

$$\lambda(\gamma,\omega) = \begin{cases} 0, & \omega = \gamma \\ 1, & \omega \neq \gamma \end{cases}$$

Mixability (1)

- consider the transformation $\mathfrak{B}_\eta : [0, +\infty]^\Omega \to [0, 1]^\Omega$ that transforms $g(\omega)$ into $e^{-\eta g(\omega)}$



- if $\mathfrak{B}_\eta(\Sigma_\Gamma)$, the image of the set of superpredictions under $\mathfrak{B}_\eta$ is convex, we call the game η-mixable
  — for an η-mixable game we can always find γ to satisfy the capital inequality

Mixability (2)

- mixability implies convexity of the set of superpredictions, but it is a bit stronger than it
  — the logarithmic and square-loss game are mixable (for some η)
  — the absolute-loss and simple prediction games are not mixable
- for an η-mixable game we can make sure that the capital inequality holds and therefore $W_t \geq p_0^{(i)} W_t^{(i)}$, i.e.,

$$\mathrm{Loss}(t) \leq \mathrm{Loss}_{\mathcal{E}_i}(t) + \frac{1}{\eta} \ln(1/p_0^{(i)})$$

(c, η)-realisability (1)

- if the game is not η-mixable, the convex hull $\mathcal{H}(\mathfrak{B}_\eta(\Sigma_\Gamma))$ is not a subset of $\mathfrak{B}_\eta(\Sigma_\Gamma)$ and $\mathfrak{B}_\eta^{-1}(\mathcal{H}(\mathfrak{B}_\eta(\Sigma_\Gamma))) \nsubseteq \Sigma_\Gamma$
- the set of superpredictions is defined in such a way that for $c \geq 1$ we have $\Sigma_\eta \subseteq \frac{1}{c}\Sigma_\eta$, i.e., $\frac{1}{c}\Sigma_\eta$ is 'larger'
- so it may still be possible that $\mathfrak{B}_\eta^{-1}(\mathcal{H}(\mathfrak{B}_\eta(\Sigma_\Gamma))) \subseteq \frac{1}{c}\Sigma_\Gamma$ — in this case we say that the aggregating algorithm is (c, η)-realisable

(c,η)-realisability (2)

- if the aggregating algorithm is (c,η)-realisable, we can find γ such that

$$\lambda(\gamma,\omega) \le -\frac{c}{\eta}\ln\sum_{i=1}^{N} p_0^{(i)} e^{-\eta\lambda(\gamma^{(i)},\omega)}$$

  for all $\omega \in \Omega$
- instead of the capital inequality, we will be able to achieve

$$\left(\frac{W_t}{W_{t-1}}\right)^{1/c} \ge \frac{\widetilde{W}_t}{\widetilde{W}_{t-1}}$$

  — let us call this the generalised capital inequality
- we get therefore $(W_t)^{1/c} \ge p_0^{(i)} W_t^{(i)}$, i.e.,

$$\mathrm{Loss}(t) \le c\,\mathrm{Loss}_{\mathcal{E}_i}(t) + \frac{c}{\eta}\ln(1/p_0^{(i)})$$

Optimality of the Aggregating Algorithm

- let us fix the uniform initial distribution $p_0^{(i)} = 1/N$
- if the aggregating algorithm is (c, η)-realisable, than for all t, all outcomes and all experts' predictions it achieves

$$\mathrm{Loss}(t) \le c\,\mathrm{Loss}_{\mathcal{E}_i}(t) + \frac{c}{\eta} \ln N$$

  for each expert $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N$
- suppose that some merging strategy for all N, all t, all outcomes and all experts' predictions achieves

$$\mathrm{Loss}(t) \le a\,\mathrm{Loss}_{\mathcal{E}_i}(t) + b \ln N$$

  for each expert $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N$ and some constants a and b — then the aggregating algorithm can do the same

1. Laissez-Faire Investment

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Capital Inequality Revisited (1)

- the generalised capital inequality

$$\left(\frac{W_t}{W_{t-1}}\right)^{1/c} \geq \frac{\widetilde{W_t}}{\widetilde{W_{t-1}}}$$

can be rewritten as

$$\frac{\widetilde{W_t}}{W_t^{1/c}} \leq \frac{\widetilde{W_{t-1}}}{W_{t-1}^{1/c}}$$

Capital Inequality Revisited (2)

- we have

$$\frac{\widetilde{W}_t}{W_t^{1/c}} = \sum_{i=1}^{N} p_0^{(i)} \frac{W_t^{(i)}}{W_t^{1/c}} = \sum_{i=1}^{N} p_0^{(i)} e^{\eta\left(\frac{\mathrm{Loss}(t)}{c} - \mathrm{Loss}_{\mathcal{E}_i}(t)\right)}$$

- if $Q_t^{(i)} = e^{\eta\left(\frac{\mathrm{Loss}(t)}{c} - \mathrm{Loss}_{\mathcal{E}_i}(t)\right)}$, the generalised capital inequality is equivalent to

$$\sum_{i=1}^{N} p_0^{(i)} Q_t^{(i)} \le \sum_{i=1}^{N} p_0^{(i)} Q_{t-1}^{(i)}$$

- this has important consequences...

Supermartingales

- let $\mathcal{P}(\Omega)$ be the set of probability distributions on Ω
  — from now on we consider finite spaces Ω, so $\mathcal{P}(\Omega)$ is the $(|\Omega|-1)$-simplex in $\mathbb{R}^{|\Omega|}$
- let E be some set of parameters
- the function $S : (E \times \mathcal{P}(\Omega) \times \Omega)^* \to \mathbb{R}$ is a supermartingale if

$$\sum_{\omega \in \Omega} \pi_t(\omega) S(e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1}, e_t, \pi_t, \omega) \leq S(e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1})$$

  for all $e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1}, e_t, \pi_t$
  — the left-hand side is the expectation of S given $e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1}, e_t, \pi_t$

Levin's Lemma

- let S be a supermartingale and let S be forecast-continuous
  — i.e., $S(e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1}, e_t, \pi, \omega_t)$ is continuous over π for all values of other parameters including t
- then for all $e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1}, e_t$ there is π such that

$$S(e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1}, e_t, \pi, \omega) \le S(e_1, \pi_1, \omega_1, \dots, e_{t-1}, \pi_{t-1}, \omega_{t-1})$$

  for all ω
- that is, we can always choose $\pi \in \mathcal{P}(\Omega)$ such that S does not grow no matter what ω the nature chooses

Applying Levin's Lemma (1)

- Levin's lemma guarantees the existence of a distribution — and we need a prediction
- if two games $\langle\Omega, \Gamma_1, \lambda_1\rangle$ and $\langle\Omega, \Gamma_2, \lambda_2\rangle$ specify the same set of superpredictions Σ, they are essentially equivalent — different Γ's and λ's can be thought of as different parametrisations of Σ
- for many games with finite Ω we can construct an equivalent parametrisation with the prediction set $\mathcal{P}(\Omega)$ — if $\Omega = \{0, 1\}$, the set $\mathcal{P}(\Omega)$ can be identified with $[0, 1]$ (cf. our definition of a binary game)
- consider $Q^{(i)} = e^{\eta\left(\frac{\mathrm{Loss}(t)}{c} - \mathrm{Loss}_{\mathcal{E}_i}(t)\right)}$ and $Q = \sum_{i=1}^{N} p_0^{(i)} Q^{(i)}$ — the array of experts' predictions $\pi_t^{(1)}, \pi_t^{(2)}, \ldots, \pi_t^{(N)}$ makes e_t

Applying Levin's Lemma (2)

- if Q is a forecast-continuous supermartingale for some c and η, on each step we can find π_t ensuring that Q does not grow
  — i.e., the generalised wealth equation is satisfied and we achieve
  $$\mathrm{Loss}(t) \leq c\,\mathrm{Loss}_{\mathcal{E}_i}(t) + \frac{c}{\eta}\ln(1/p_0^{(i)})$$
  — we shall call this method defensive forecasting
- if Q is a forecast-continuous supermartingale for some c and η, the aggregating algorithm must be (c, η)-realisable and output the same predictions
  — the inverse statement is a bit trickier...

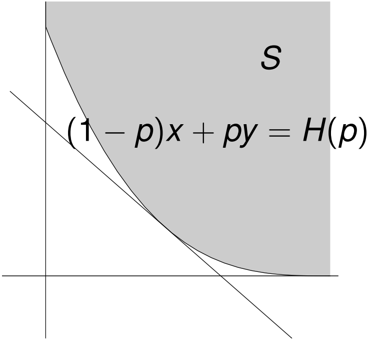
Proper Loss Function

- a function $\lambda : \mathcal{P}(\Omega) \times \Omega \to [0, +\infty$ is proper if for all $\pi, \pi' \in \mathcal{P}(\Omega)$ we have

$$\mathbf{E}_{\omega \sim \pi} \lambda(\pi, \omega) \leq \mathbf{E}_{\omega \sim \pi} \lambda(\pi', \omega)$$

- the expectation of $\lambda(\pi', \omega)$ when ω is distributed according to π is the smallest when $\pi' = \pi$
- a proper and continuous loss function exists given some mild technical restrictions on the game
  — the square and logarithmic loss functions are already proper

Geometric Interpretation



- a distribution π is a family of parallel hyperplanes
- for a proper loss function λ this family touches S at $(\lambda(\omega_0, \pi), \lambda(\omega_1, \pi), \ldots, \lambda(\omega_{K-1}, \pi))$

The Inverse Statement

- if AA is (c, η)-realisable and there is a proper continuous loss function, then Q is a supermartingale
- and defensive forecasting achieves the same loss bound as the aggregating algorithm

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