Tailoring the Tails: Enhancing the Reliability of Probabilistic Load Forecasts

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a joint work with **Roberto Baviera**, *Politecnico di Milano*

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Quantitative Finance LAB

COPA 2024 Conference

Milan, 10 September 2024

Probabilistic Load Forecasting

The time series displays **seasonality on multiple time scales**

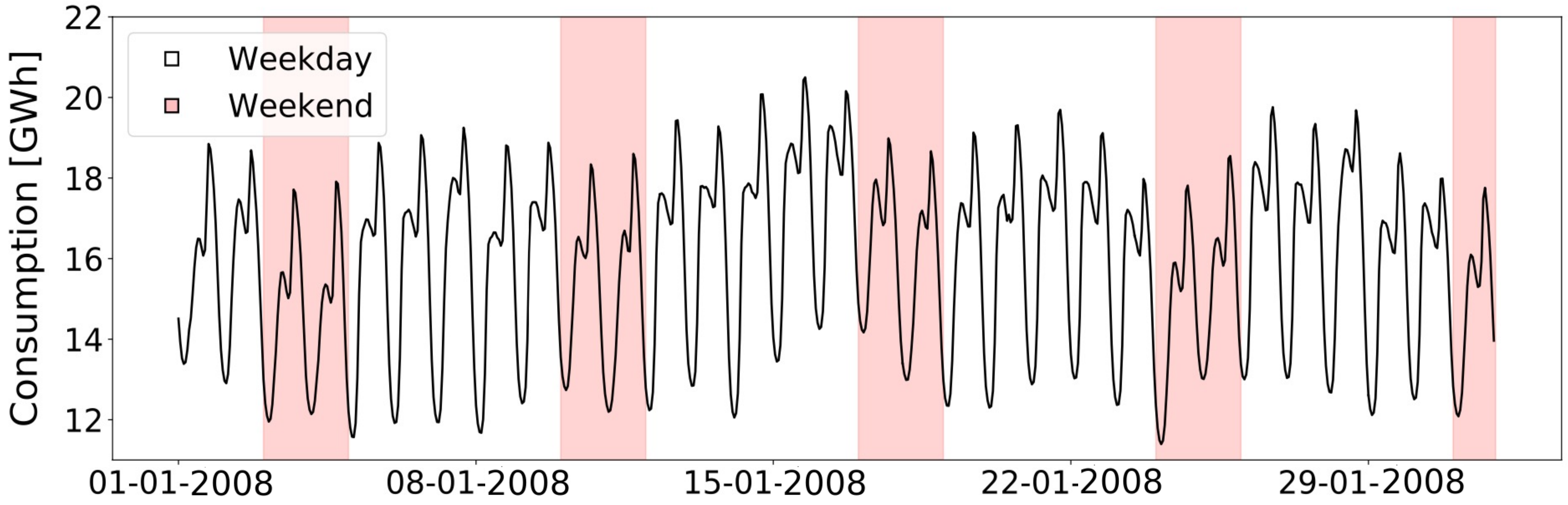

Figure: **Hourly** aggregate consumption of New England

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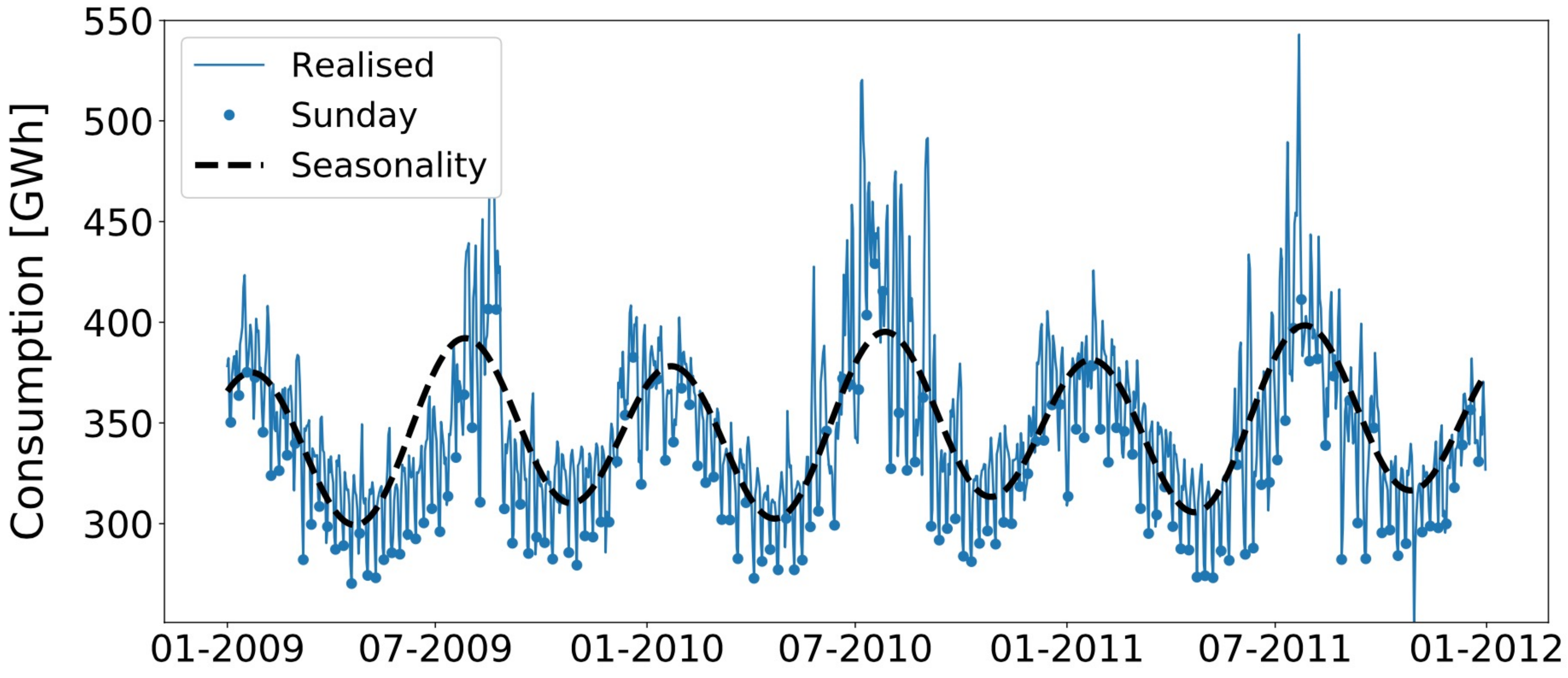

Figure: **Daily** aggregate consumption of New England

Probabilistic Load Forecasting

The time series displays evident **seasonal heteroskedasticity**

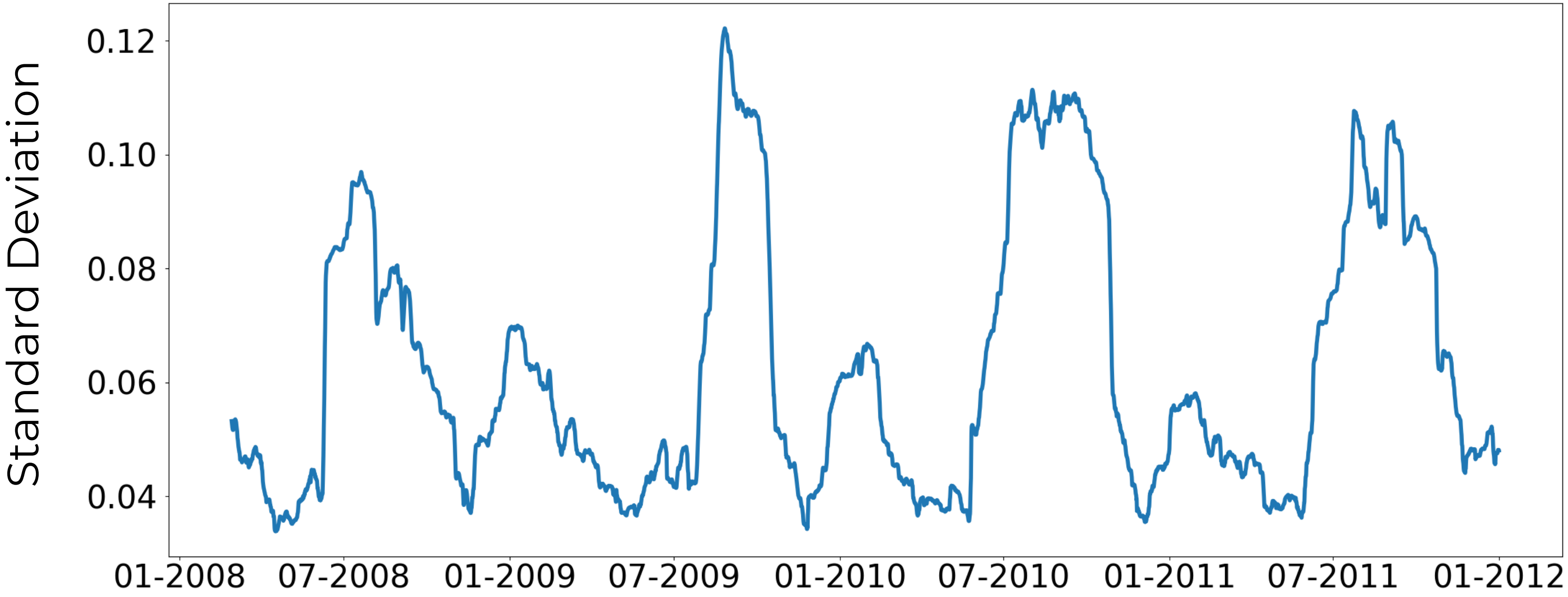

Figure: **Standard Deviation** of deseasonalised time series of consumption (on a 2-months rolling window)

Probabilistic Load Forecasting

- It plays a vital role in decision-making processes: unit commitment, reserve management, economic dispatch and maintenance scheduling

- Forecast errors may result in grid instability, significant financial losses and potential blackouts.

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It is fundamental to take into account predictive **overconfidence**

overestimate **likely** events

underestimate **unlikely** events

Research questions

How can we generate probabilistic forecasts for highly **non-**exchangeable time series data?

How can we manage **overconfidence** in probabilistic load forecasting?

What does **overconfidence** look like?

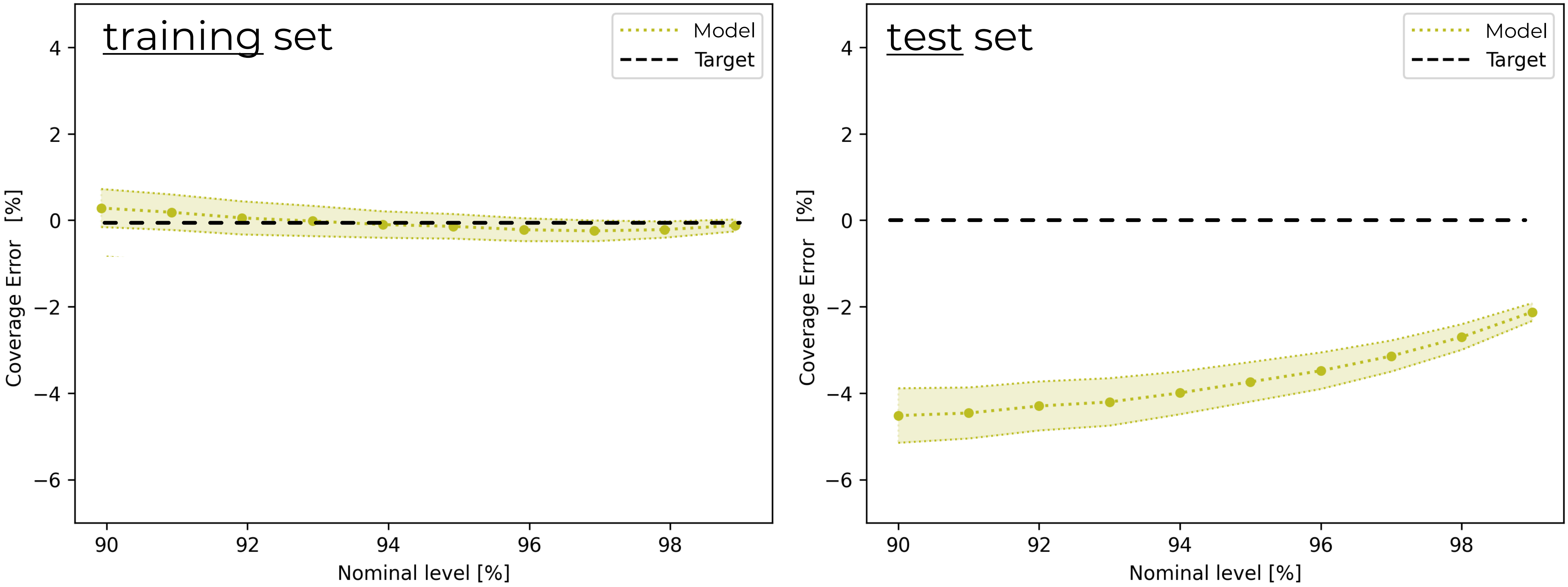

Figure: Prediction Intervals coverage plots on training and test sets
Bands are intended to represent *SE* for 10 repetition with different seed

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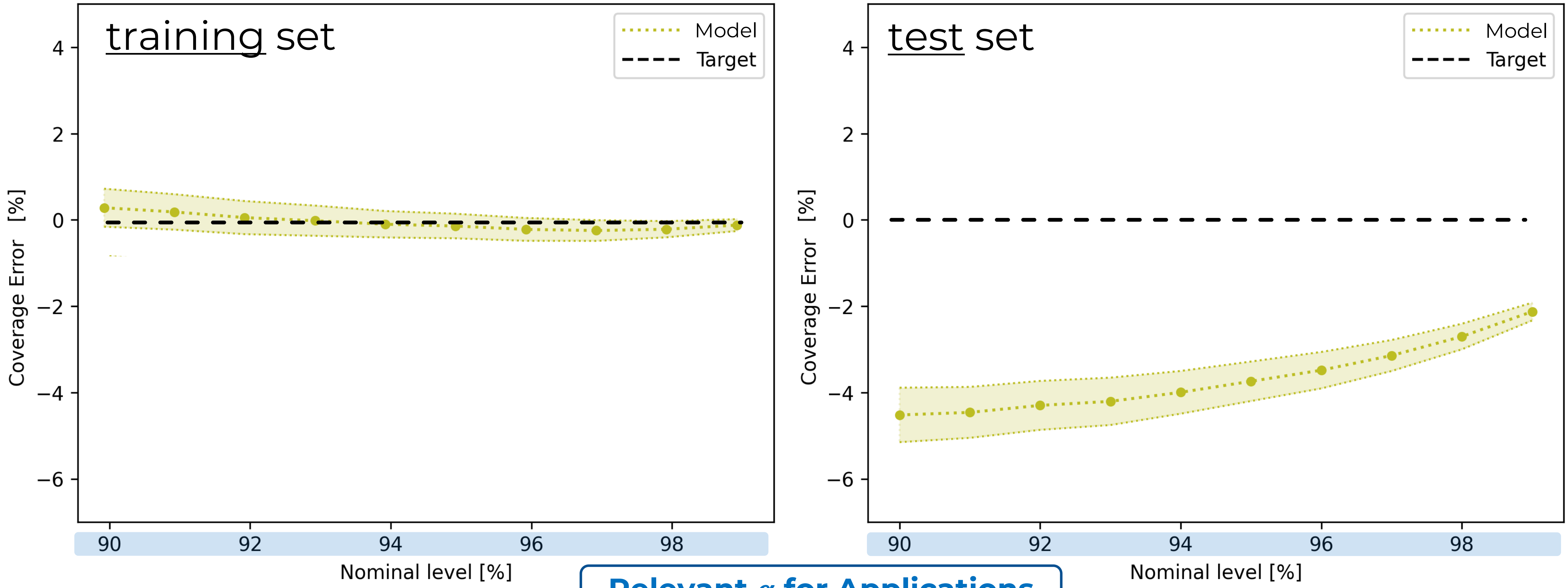

Relevant α for Applications

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Bands are intended to represent *SE* for 10 repetition with different seed

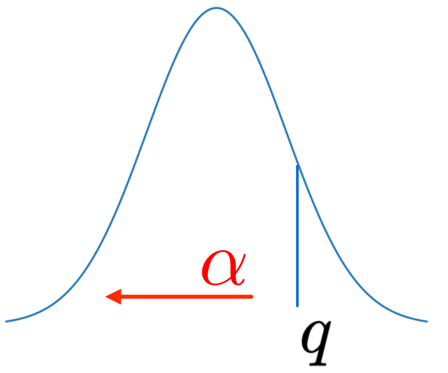
Outline of the talk

- ✓ Introduction and motivation
- ○ The proposed methodology
- ○ Results
- ○ Conclusions

Pinball Loss

- Pinball Loss (PL; Könker & Bassett, 1978) – Evaluating Single Quantiles

$$\mathcal{P}\left(q, y; \alpha\right) := \begin{cases} \alpha\ (y - q) & \textbf{if} \quad y \geq q \\ (1-\alpha)(q - y) & \textbf{if} \quad y < q \end{cases}$$



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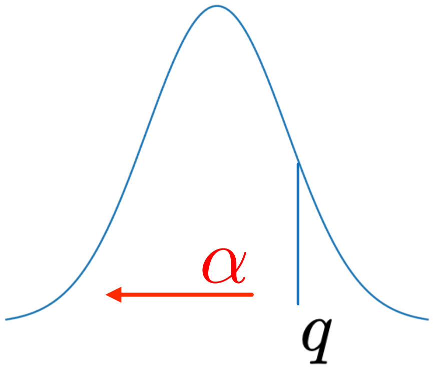
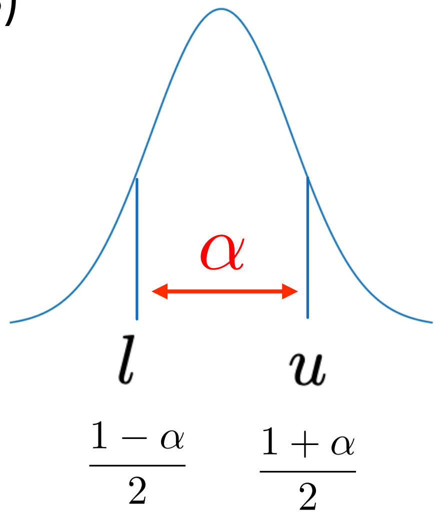


- Central Pinball Loss – Adapting PL for Prediction Intervals (PIs)

$$\mathcal{P}_{\mathrm{C}}(l, u, y;\ \alpha) := \mathcal{P}\left(l,\ y\,;\ \frac{1-\alpha}{2}\right) + \mathcal{P}\left(u,\ y\,;\ \frac{1+\alpha}{2}\right)$$



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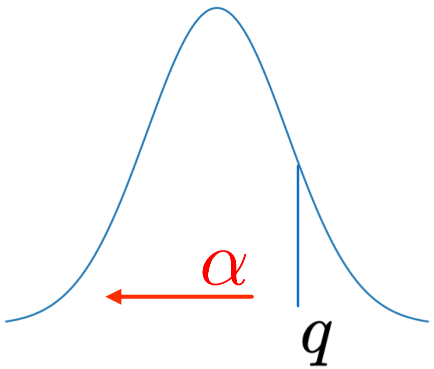

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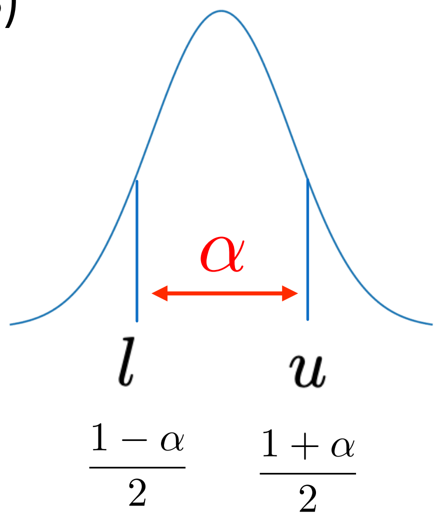
$$= \frac{1-\alpha}{2}(u-l) + \left(\,[y-u]^{+} + [l-y]^{+}\,\right)$$



From PI to Distributions: CRPS

- When a probabilistic forecasting model issues a CDF ***F***

  ⟶ average Central Pinball Loss over all confidence levels

(see, e.g., Gneiting & Raftery, 2007)

Continuous Ranked Probabilistic Score (Matheson & Winkler, 1986)

$$\mathrm{CRPS}(F, y) = \int_0^1 \mathcal{P}_{\mathrm{C}} \left(\underbrace{F^{-1}\left(\frac{1-\alpha}{2}\right), F^{-1}\left(\frac{1+\alpha}{2}\right)}_{\textbf{Extrema} \text{ of the } \boldsymbol{\alpha}\textbf{-PI}}, y \, ; \, \alpha \right) \mathrm{d}\alpha$$

- <u>Often</u> used for forecast assessment, <u>less often</u> as a loss function

Tackling predictive **overconfidence**

The goal: building a new robust loss function to be used at training time

$$\mathcal{P}_{\mathrm{C}}(l, u, y; \alpha) = \frac{1-\alpha}{2}(u-l) + \begin{cases} (y-u) & \text{if} \quad y > u\,, \\ 0 & \text{if} \quad y \in [l, u]\,, \\ (l-y) & \text{if} \quad y < l\,. \end{cases}$$

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- For any $\lambda \in [0, 1)$, we define the ***λ-adjusted*** Central Pinball Loss as

$$\mathcal{P}_{\mathrm{C}}^{[\lambda]}(l, u, y;\, \alpha) = (1-\lambda)\,\frac{1-\alpha}{2}\,(u-l) + \begin{cases} (y-u) & \text{if } \ y > u\,, \\ 0 & \text{if } \ y \in [l, u]\,, \\ (l-y) & \text{if } \ y < l\,. \end{cases}$$

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Computing the integral average of the new loss measure over quantiles:

$$\mathrm{CRPS}^{[\lambda]}(F,y) := \int_0^1 \mathcal{P}_{\mathrm{C}}^{[\lambda]}\left(F^{-1}\left(\frac{1-\alpha}{2}\right),\, F^{-1}\left(\frac{1+\alpha}{2}\right),\, y\,;\,\alpha\right)\mathrm{d}\alpha$$

- This scoring rule is well-defined under mild integrability assumptions for the CDF and can be explicity computed for all relevant distributions.

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Proposition: Let F be the CDF of a Gaussian with mean μ and variance σ^2. Then:

$$\mathcal{G}\text{-}\mathrm{CRPS}^{[\lambda]}(\mu, \sigma, y) := \sigma\left[\frac{y-\mu}{\sigma}\left(2\mathcal{N}\left(\frac{y-\mu}{\sigma}\right)-1\right)+2\varphi\left(\frac{y-\mu}{\sigma}\right)-\frac{\lambda+\sqrt{2}(1-\lambda)}{\sqrt{\pi}}\right]$$

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$$\mathcal{G}\text{-}\mathrm{CRPS}^{[\lambda]}(\mu, \sigma, y) := \mathcal{G}\text{-}\mathrm{CRPS}(\mu, \sigma, y) - \lambda\left(\frac{\sqrt{2}-1}{\sqrt{\pi}}\,\sigma\right)$$

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CORRECTION TERM

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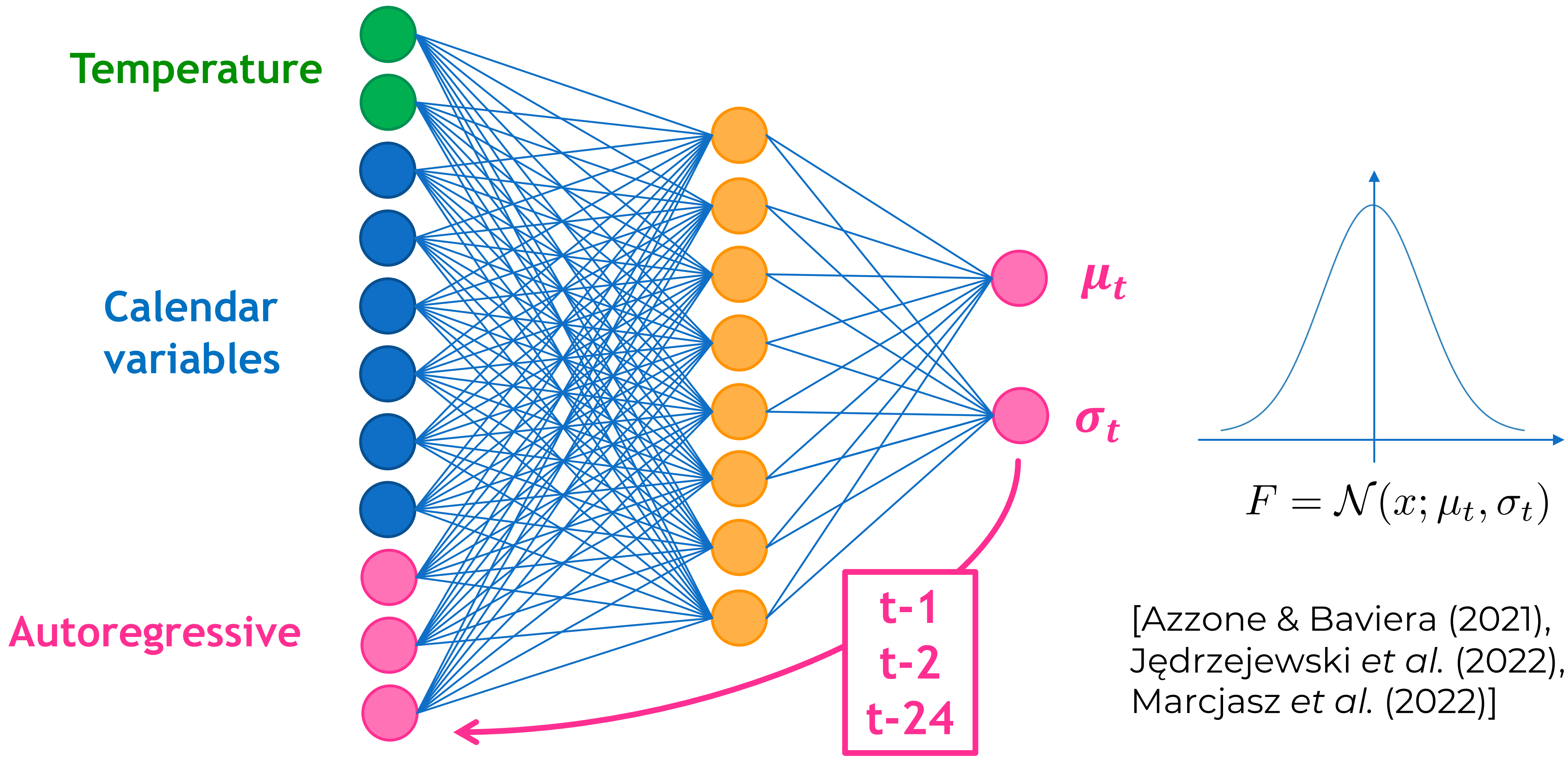
Modelling Approach



[Azzone & Baviera (2021), Jędrzejewski *et al.* (2022), Marcjasz *et al.* (2022)]

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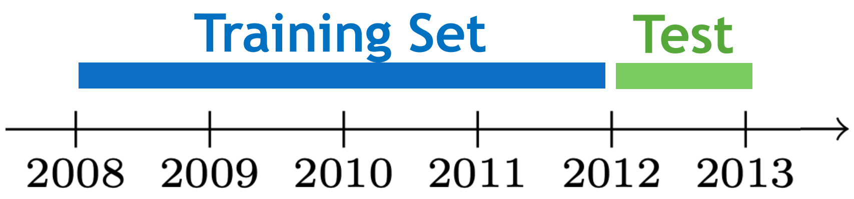

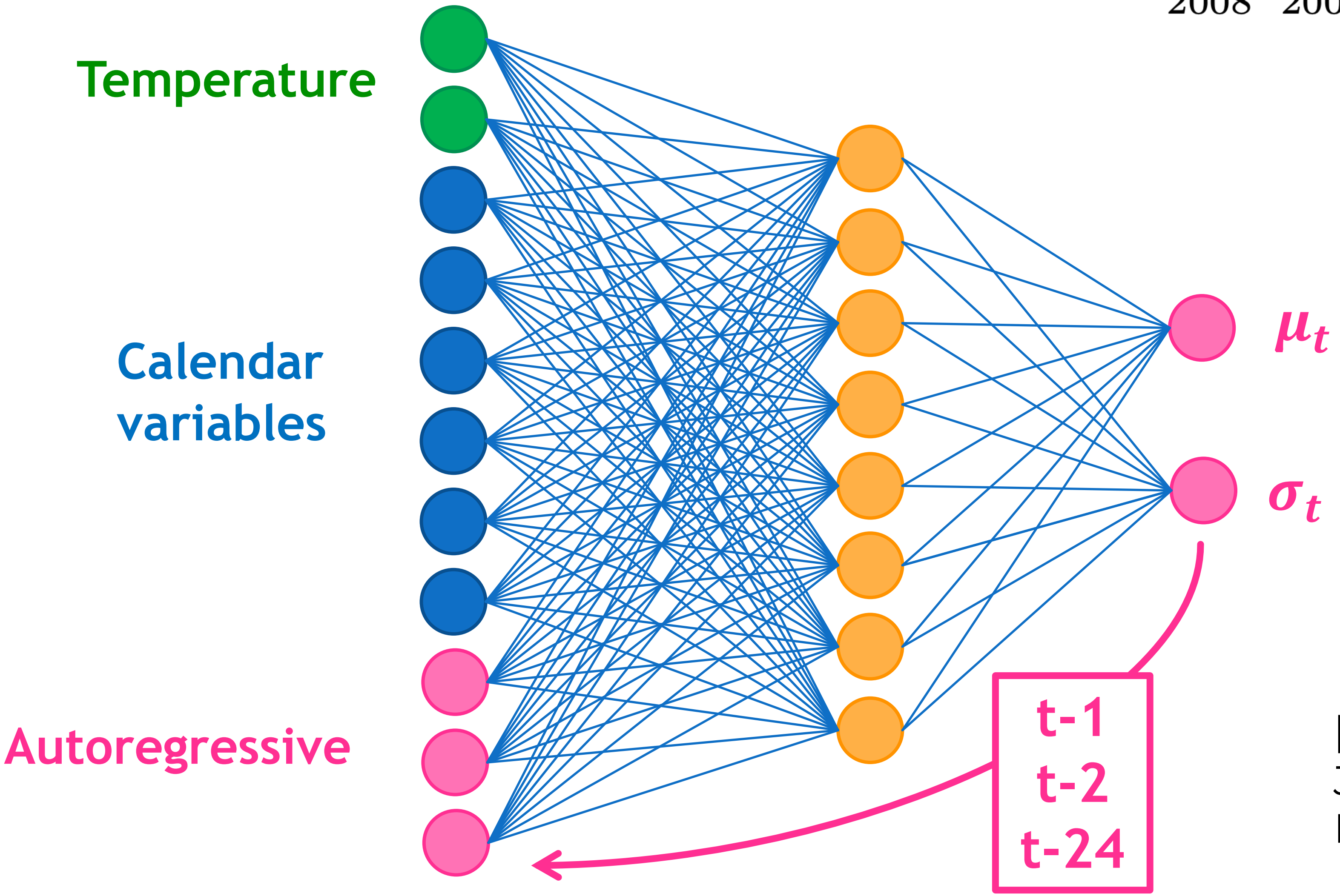

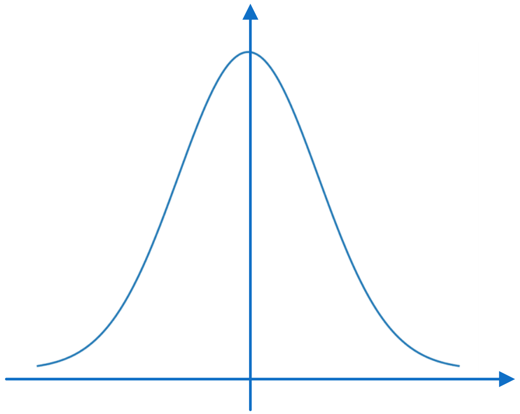

$$F = \mathcal{N}(x; \mu_t, \sigma_t)$$

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PIs Backtesting (on test set, 2012)

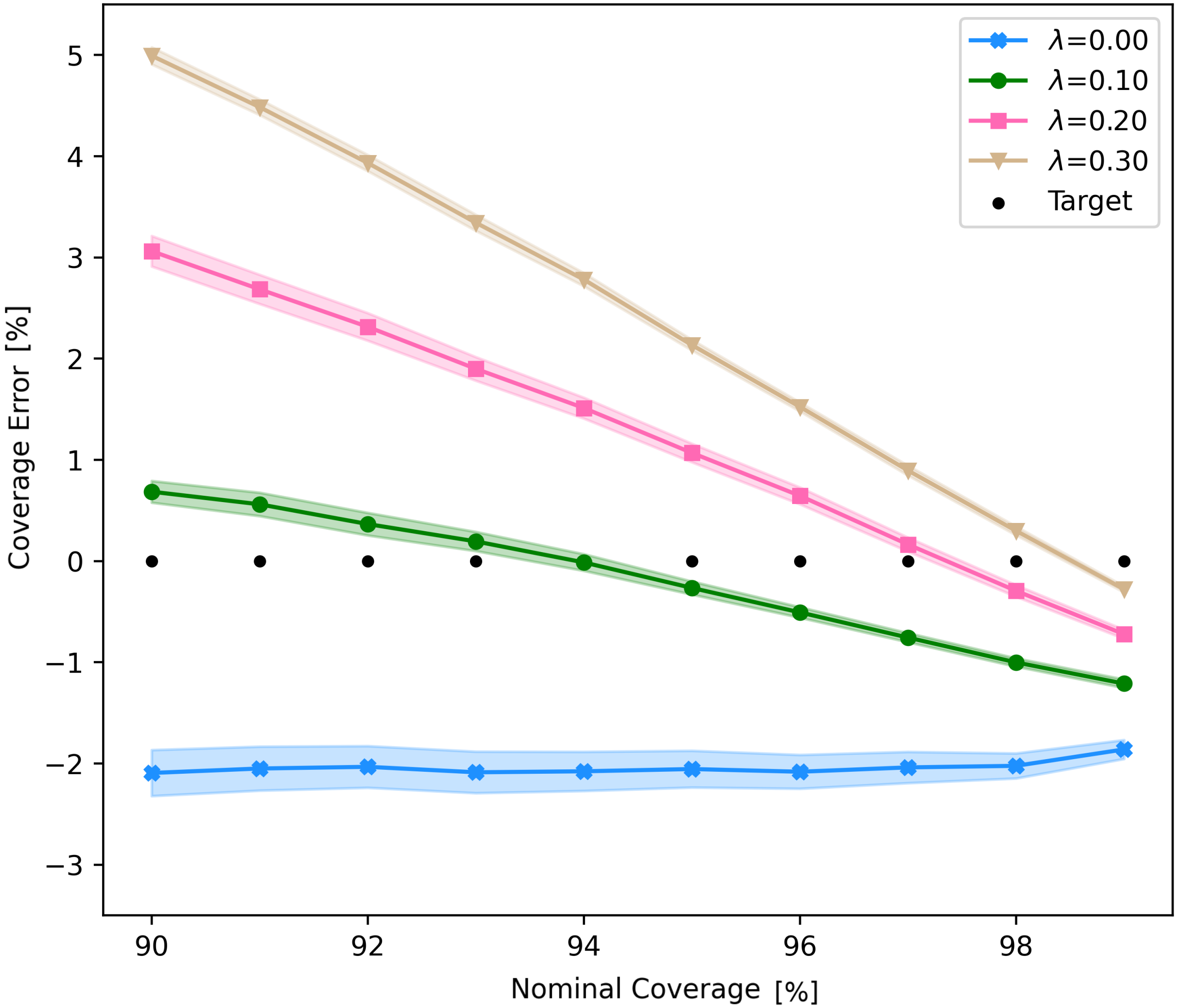

Figure: Prediction Intervals coverage plots with SE bands (10 repetitions)

Numerical Results (on test set, 2012)

$\mathcal{G}$-CRPS$^{[\lambda]}$

λ	MAPE [%]	RMSE [MWh]	CRPS [MWh]	EC(90%) [%]	EC(95%) [%]	AACE [%]
0.00	2.01 ± 0.01	409.58 ± 1.31	104.23 ± 0.33	87.90 ± 0.22	92.94 ± 0.18	2.03 ± 0.17
0.05	2.00 ± 0.01	408.45 ± 1.60	103.97 ± 0.39	89.21 ± 0.14	93.64 ± 0.17	1.26 ± 0.13
0.10	2.00 ± 0.01	406.49 ± 1.64	103.87 ± 0.43	90.68 ± 0.10	94.73 ± 0.06	0.59 ± 0.03
0.15	1.99 ± 0.01	406.11 ± 1.67	103.77 ± 0.42	91.87 ± 0.19	95.36 ± 0.11	0.91 ± 0.08
0.20	2.00 ± 0.01	406.91 ± 1.50	104.12 ± 0.40	93.06 ± 0.15	96.06 ± 0.09	1.44 ± 0.07
0.25	2.00 ± 0.01	407.78 ± 1.49	104.62 ± 0.42	94.06 ± 0.11	96.60 ± 0.07	1.94 ± 0.06
0.30	2.00 ± 0.01	408.67 ± 1.76	105.27 ± 0.46	94.99 ± 0.08	97.13 ± 0.05	2.46 ± 0.05

Table: Results on the test set (2012)

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$\mathcal{G}$-MLE

MAPE [%]	RMSE [MWh]	CRPS [MWh]	EC(90%) [%]	EC(95%) [%]	AACE [%]
2.05 ± 0.01	423.59 ± 1.20	107.54 ± 0.39	89.28 ± 0.26	93.70 ± 0.19	1.33 ± 3.87

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Numerical Results (on test set, 2012)

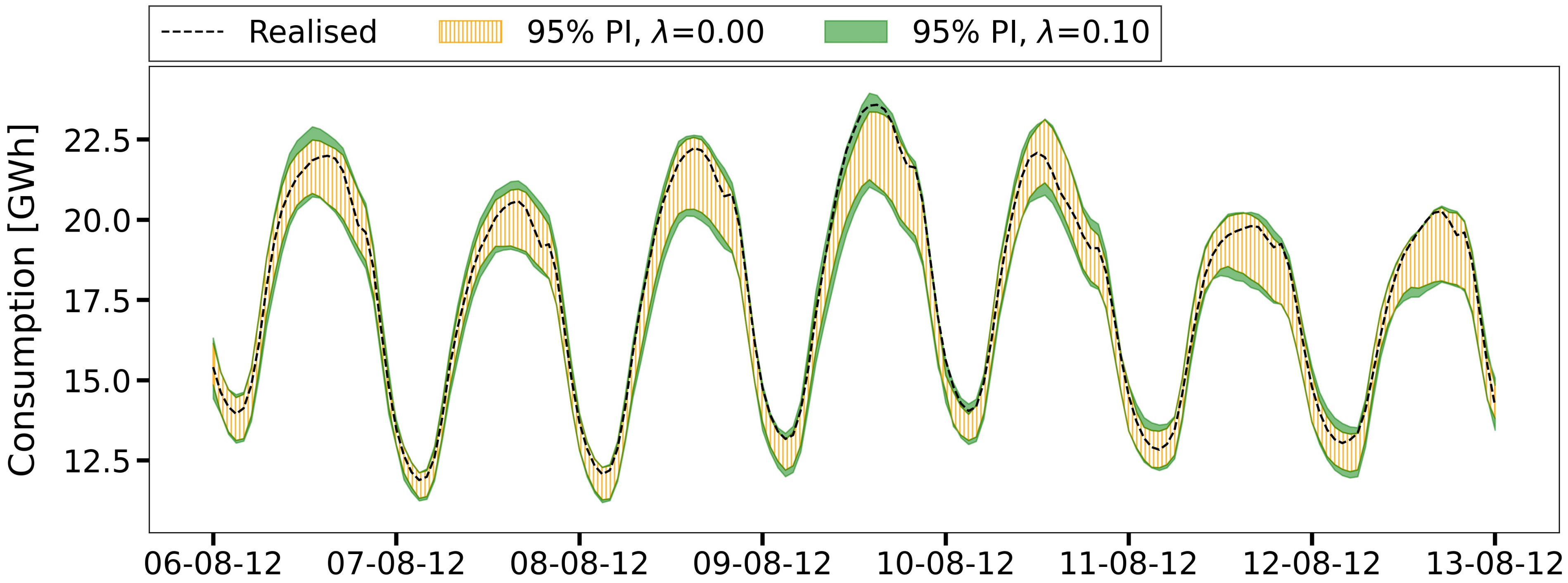

Figure: Modification of density forecasts of hourly demand on the test set (2012) for a week of August, when $\mathcal{G}$ -CRPS$^{[0.00]}$, the original CRPS loss function, and $\mathcal{G}$ -CRPS$^{[0.10]}$, the new proposed version of the CRPS, are used to train the predictive model.

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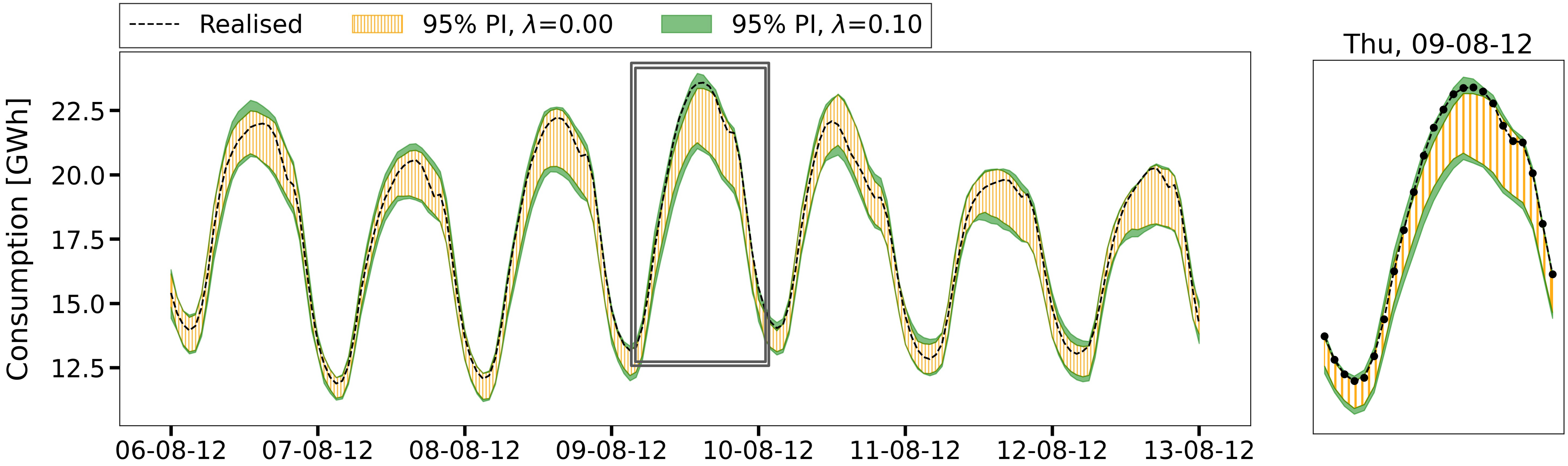


Figure: Modification of density forecasts of hourly demand on the test set (2012) for a week of August, when $\mathcal{G}$ -CRPS$^{[0.00]}$, the original CRPS loss function, and $\mathcal{G}$ -CRPS$^{[0.10]}$, the new proposed version of the CRPS, are used to train the predictive model.

Numerical Results

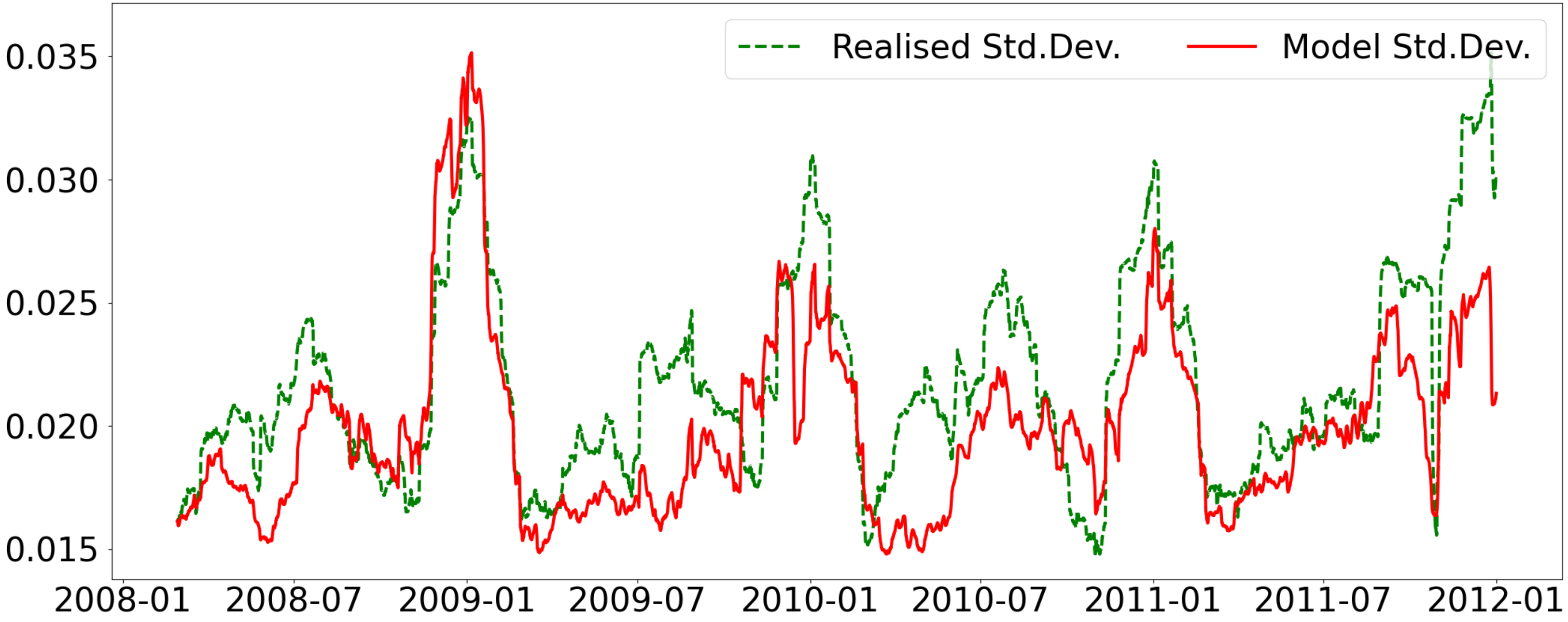
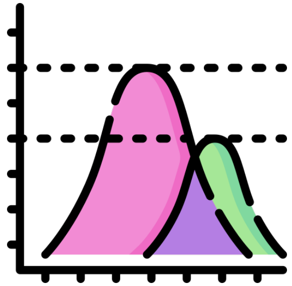


Figure: Realised *standard deviation* and *standard deviation* fitted by the RNN model when $\mathcal{G}$ -CRPS$^{[0.10]}$ is used to train the predictive model. Standard deviation is computed considering a 2-months rolling-window approach.

Conclusions

i) We have designed a new parsimonious forecasting methodology for time series that display complex patterns (heteroskedasticity, autocorrelation, multi-seasonality)



ii) We have designed a new family of loss functions to tackle predictive **overconfidence** in probabilistic time series forecasting

iii) We have tested the methodology on a benchmark dataset. The predictive performance is **excellent** both in terms of **reliability** and point **accuracy**

Essential bibliography

- Azzone, M. and Baviera, R. (2021). Neural network middle-term probabilistic forecasting of daily power consumption, *Journal of Energy Markets*, 1 (14), 1–26.
- Benth, F. E., Benth, J. S., & Koekebakker, S. (2008). *Stochastic modelling of electricity and related markets*. World Scientific.
- Berrisch, J., and Ziel, F. (2023). CRPS learning. *Journal of Econometrics*, 237:105221.
- Jędrzejewski, A., Lago, J., Marcjasz, G., & Weron, R. (2022). Electricity Price Forecasting: The Dawn of Machine Learning. *IEEE Power and Energy Magazine*, *20*(3), 24-31.
- Gneiting, T. & Raftery, A. E. (2007). Strictly proper scoring rules, prediction, and estimation. *Journal of the American statistical Association*, 102(477):359–378.
- Hong, T., Pinson, P., Fan, S., Zareipour, H., Troccoli, A., and Hyndman, R.J. (2016). Probabilistic energy forecasting: Global Energy Forecasting Competition 2014 and beyond. *International Journal of Forecasting*, 32(3):896–913.
- Marcjasz, G., Narajewski, M., Weron, R., & Ziel, F. (2022). Distributional neural networks for electricity price forecasting. *arXiv preprint arXiv:2207.02832*.
- Matheson, J.E., and Winkler, R.L. (1976) Scoring rules for continuous probability distributions. *Management Science*, 22(10):1087–1096