Black–Scholes formula without stochastic assumptions

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Abstract

The usual stochastic assumptions of the Black–Scholes formula are replaced by assumptions about fractal dimensions describing the price paths of the underlying security and the *square* (the European option which pays the squared price of the underlying at the expiry). In particular, it is sufficient to require that the price path of the underlying security should have a fractal dimension of at most $1\frac{1}{2}$ (which is the fractal dimension of typical stochastic processes, such as the price path of the underlying in the Black–Scholes model) and that the "remaining volatility path" implied by the square should have a fractal dimension of less than $1\frac{1}{2}$ (ie, it should be "substochastic"). For simplicity we assume zero interest rate.

1 Introduction

Consider a European option with payoff $U(S(T))$, where $S(T)$ is the price of the underlying security at the expiry T. According to the Black–Scholes formula, the fair price for $U(S(T))$ at time t is

$$U(t) = \int_{\mathbf{R}} U(S(t)e^z)\ \mathcal{N}_{-D(t)/2, D(t)}(dz), \tag{1}$$

where $S(t)$ is the current price of the underlying, $D(t) = \sigma^2(T-t)$ is the *remaining volatility*, and $\mathcal{N}_{\mu,d}$ is the normal distribution with mean μ and variance d. In particular, the initial fair price is obtained by setting $t = 0$.

The Black–Scholes approach to valuing options is standard, but still it is widely recognized that the stochastic differential equation (the "Black–Scholes model")

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \tag{2}$$

on which it is based, is an oversimplification.

Another standard way of pricing $U(S(T))$ is to approximate the payoff function U by a combination of call and put options and use the market prices of calls and puts to find the fair price of $U(S(T))$. This approach does not rely on any stochastic assumptions about $S(t)$, but its weak side is that to obtain the desired payoff function, we might need many calls and puts with different strike prices; even assuming that all these calls and puts are traded, all of them will have bid/ask spreads, which might be accumulated to make the accuracy of the computed price too low.

In this paper we suggest a third approach (to some degree intermediate between the previous two): instead of the infinite (in practice, large) sets of derivatives P_a (put with strike a) and C_a (call with strike a), we price $U(S(T))$ with just one European option V. To do so, we will still need to make some assumptions about the behaviour of the market prices, but these assumptions will be weaker than the Black–Scholes model (2).

Remark 1 This paper develops the approach of [3] and [4], in which we assumed that, instead of a European option V, the security which pays a dividend of $\left(\frac{dS(t)}{S(t)}\right)^2$ (until the expiry time T) is traded. (If such a security is traded, its current price at time t can replace the remaining volatility $D(t)$.)

2 Definitions

According to the Black–Scholes equation (1), the fair price at time t of the European option $V(S(T))$ is

$$V(t) = \int_{\mathbf{R}} V(S(t)e^z)\ \mathcal{N}_{-D(t)/2, D(t)}(dz).$$

If we know the price paths $S(t)$ and $V(t)$ for the underlying security and the option, we can find, under some conditions of regularity, the *implied remaining volatility path* $D(t)$. In this paper we will only consider the simplest case where V is the *square*, ie the European option which pays $S^2(T)$ at the expiry time T. Let $V(t)$ be the price path of the square; it is clear that $V(T) = S^2(T)$. For the square we obtain the Black–Scholes price of

$$V(t) = \int_{\mathbf{R}} (S(t)e^z)^2 \; \mathcal{N}_{-D(t)/2,\,D(t)}(dz) = S^2(t)e^{D(t)}$$

at time t; therefore, the implied remaining volatility path is $D(t) = \ln \frac{V(t)}{S^2(t)}$.

Remark 2 Notice that in the Black–Scholes model the implied remaining volatility path is linear:

$$D(t) = \sigma^2(T-t).$$

In this extended abstract we will only consider pricing European options for which the payoff function is smooth and has bounded derivatives of all orders. This is easy to generalize, since typical payoff functions (such as those for calls and puts) can be arbitrarily well approximated with smooth functions with bounded derivatives.

Our assumption will be that the fractal dimension of $S(t)$ and $D(t)$ is not too high; in this section we will define what we mean by "fractal dimension". We will use notions of non-standard analysis such as "infinitesimal" and "infinitely large"; those are just analytical explications of "very small" and "very large".

The time interval $[0,T]$ is divided into Ω equal infinitely short subintervals; here Ω is an infinitely large positive integer. We put

$$dt := \frac{T}{\Omega}, \; \mathbf{T} := \{idt \mid 0 \le i \le \Omega\},$$

where i ranges over the nonnegative integers, maybe infinitely large.

First of all, we assume that the paths $S(t)$ and $D(t)$ are continuous; the notion of fractal dimension will be defined only for continuous functions. For each continuous function $f\colon [0,T] \to \mathbf{R}$ and each constant $c \in [1,\infty[$, we define the *c-variation* $\mathbf{var}_c\, f \ge 0$ to be the sum

$$\sum_{t \in \mathbf{T}\setminus\{T\}} |f(t+dt) - f(t)|^c.$$

(For $c = 2$, this corresponds to the familiar *quadratic variation* in the theory of stochastic processes.)

Lemma 1 *For each continuous function* $f\colon [0,T] \to \mathbf{R}$ *there exists a unique number* $\operatorname{vex} f \in [1,\infty]$ *(the* volatility exponent *of* f*) such that*

- $\mathbf{var}_c\, f$ *is infinitely large when* $1 \le c < \operatorname{vex} f$*;*

◇ $\mathbf{var}_c f$ *is infinitesimal when* $c > \operatorname{vex} f$.

To understand the intuition behind this notion, suppose that the increments $f(t+dt) - f(t)$ of f have the order of magnitude $(dt)^H$. (For the "usual" deterministic functions like $\sin t$ we have $H = 1$; for the typical paths of "regular" diffusion processes with positive diffusion coefficient, such as (2), we have $H = \frac{1}{2}$.) In this case, $\mathbf{var}_c f$ has the order of magnitude $(dt)^{cH-1}$, and we obtain $\operatorname{vex} f = \frac{1}{H}$. Reversing this argument, we define the *Hölder exponent* of f to be $H(f) := \frac{1}{\operatorname{vex} f}$. The value $\dim f := 2 - H(f)$ will be called the *fractal dimension* (of the graph) of f. (This is a variation of well-known definitions; see, eg, [1].) For "deterministic" functions like sin the fractal dimension is 1 (their graphs are 1D lines); for "stochastic" functions such as typical paths of (2) the fractal dimension is $1\frac{1}{2}$ (their graphs are halfway between 1D lines and the 2D plane).

3 Theorem

The next theorem gives the fair price of a smooth European option U in terms of the square V.

Theorem 1 *Let* $U\colon \mathbf{R} \to \mathbf{R}$ *be a smooth European option with all derivatives bounded and* V *be the square. The fair price at time 0 for* $U(S(T))$ *is*

$$\int_{\mathbf{R}} U(S(0)e^z)\ \mathcal{N}_{-D(0)/2,D(0)}(dz), \tag{3}$$

where $D(t) = \ln \frac{V(t)}{S^2(t)}$ *is the remaining volatility path implied by* V, *provided that* $D(t)$ *and* $S(t)$ *are continuous,* $S(t) > 0$ *for all* t, $D(T) = S^2(T)$ *and*

$$\dim D < 1\frac{1}{2},\ \dim S < 1\frac{2}{3},\ \dim D + \dim S < 3. \tag{4}$$

In this extended abstract we have not given a formal definition of a fair price, but it should be clear what the statement of the theorem means: starting with a capital of (3), we can manage our portfolio containing shares S and squares V in such a way that its final value is infinitely close to $U(S_T)$, provided that (4) is satisfied.

There are two important special cases of (4). The first case was already mentioned in the abstract: $\dim D < 1\frac{1}{2}, \dim S \le 1\frac{1}{2}$. The second case is $\dim D \le 1\frac{1}{3}, \dim S < 1\frac{2}{3}$. In the Black–Scholes model we have $\dim D = 1$ (see Remark 2) and $\dim S = 1\frac{1}{2}$, which is covered by both special cases.

4 Proof sketch

We will find a trading strategy which starts with initial capital (3) and ensures that the capital at time t is infinitely close to

$$\int_{\mathbf{R}} U(S(t)e^z)\ \mathcal{N}_{-D(t)/2,D(t)}(dz).$$

For any $D \ge 0$ and $S > 0$, we put

$$\overline{U}(D,S) := \int_{\mathbf{R}} V(Se^z)\ \mathcal{N}_{-D/2,D}(dz);$$

a similar notation will be used for V. The Black–Scholes (or Kolmogorov's) partial differential equation is

$$\frac{\partial \overline{U}}{\partial D} = \frac{1}{2} S^2 \frac{\partial^2 \overline{U}}{\partial S^2}.$$

Combining this with Taylor's formula, we obtain (making use of the notation $df(t) := f(t+dt) - f(t)$)

$$\begin{aligned} d\overline{U}(D(t),S(t)) &\approx \frac{\partial \overline{U}}{\partial D}(D(t),S(t))dD(t) + \frac{\partial \overline{U}}{\partial S}(D(t),S(t))dS(t) \\ &\quad + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial S^2}(D(t),S(t))(dS(t))^2 \\ &= \frac{\partial \overline{U}}{\partial D}(D(t),S(t))\left(dD(t) + \left(\frac{dS(t)}{S(t)}\right)^2\right) + \frac{\partial \overline{U}}{\partial S}(D(t),S(t))dS(t). \end{aligned} \tag{5}$$

From the analogous equality for V, we find:

$$dD(t) + \left(\frac{dS(t)}{S(t)}\right)^2 \approx \frac{1}{\frac{\partial \overline{V}}{\partial D}(D(t),S(t))} dV(t) - \frac{\frac{\partial \overline{V}}{\partial S}(D(t),S(t))}{\frac{\partial \overline{V}}{\partial D}(D(t),S(t))} dS(t). \tag{6}$$

Substituting (6) into (5) and summing over $t \in \mathbf{T} \setminus \{T\}$, we obtain that our goal will be attained by the trading strategy

$$\left.\begin{array}{r} \dfrac{\frac{\partial \overline{U}}{\partial D}(D(t),S(t))}{\frac{\partial \overline{V}}{\partial D}(D(t),S(t))} \text{ units of } V \\ \frac{\partial \overline{U}}{\partial S}(D(t),S(t)) - \dfrac{\frac{\partial \overline{U}}{\partial D}(D(t),S(t))\frac{\partial \overline{V}}{\partial S}(D(t),S(t))}{\frac{\partial \overline{V}}{\partial D}(D(t),S(t))} \text{ units of } S \end{array}\right\} \text{ at time } t,$$

provided that our approximations (implicit in using the sign "$\approx$" above) are not too crude.

It is clear that the accumulated error has the order of magnitude

$$\begin{aligned} &\mathbf{var}_2\, D + \mathbf{var}_3\, S + \sum_{t \in \mathbf{T}\setminus\{T\}} |dD(t)||dS(t)| \\ &\le \mathbf{var}_2\, D + \mathbf{var}_3\, S + (\mathbf{var}_p\, D)^{1/p} (\mathbf{var}_q\, S)^{1/q}, \end{aligned}$$

where $p, q \in \,]1,\infty[$ are chosen to satisfy $\operatorname{vex} D < p$, $\operatorname{vex} S < q$ and $\frac{1}{p} + \frac{1}{q} = 1$. We can see that our assumptions (4) imply that the accumulated error will be infinitesimal.

5 Discussion

In this paper we have described an approach to dynamic hedging which can potentially improve both the reliability of hedging strategies and market liquidity. Trading in squares is a direct way of estimating the future volatility of the underlying security, which makes hedging less dependent on our beliefs about the market (such as the Black–Scholes model (2)). A range of squares with different maturities allow pricing all European options, which makes a market based on squares potentially more liquid than that based on the double-indexed ranges of calls and puts.

In the rest of this section we will discuss possible directions of future research. (Some of these suggestions are implemented in [3] for the approach described in Remark 1.)

The most obvious direction is to extend our approach to other kinds of options such as American. We do not expect that pricing American options will present any difficulties besides the usual ones.

Our assumption of zero interest rate is unrealistic. It is easy to replace it with the assumption of known constant interest rate, but this new assumption is no more realistic. Probably the most natural approach is to allow our portfolio to contain Treasury bills with expiry time T. The initial price of the Treasury bill will enter the Black–Scholes formula (1) and its fractal dimension (and maybe also the fractal dimension of some derivative based on the interest rate) will enter the conditions (4).

The exposition based on non-standard analysis is simple and mathematically elegant. However, in practical applications it is important to have explicit upper bounds for the approximation error of the hedging strategies. For such upper bounds the fractal dimension might turn out to be too crude a tool: in practice we will not have the theoretical nice separation (see Lemma 1) of the c for which $\mathbf{var}_c$ is small from the c for which $\mathbf{var}_c$ is large; we expect that most useful bounds will be given directly in terms of the variations.

And, of course, there is the problem of empirical measurements of the fractal dimension of shares and squares. Preliminary experiments show that typically fractal dimensions for shares lie in the region 1.3–1.5, but occasionally they exceed 1.5. (Under our definition: of course, dim is not the only possible explication of fractal dimension. For example, Hurst exponents have been widely used in finance: see, eg, [2].) I am not aware of any squares traded at the moment, so checking our assumptions (4) might be problematic. If our assumptions are falsified, we can always look for new "fractal models" (of the type (4)) rather than usual "stochastic models" (such as (2)).

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